

# Linear transformations that are tridiagonal with respect to the three decompositions for an LR triple

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## Abstract

Recently, Paul Terwilliger introduced the notion of a lowering-raising (or LR) triple, and classified the LR triples. An LR triple is defined as follows. Fix an integer  $d \geq 0$ , a field  $\mathbb{F}$ , and a vector space  $V$  over  $\mathbb{F}$  with dimension  $d + 1$ . By a decomposition of  $V$  we mean a sequence  $\{V_i\}_{i=0}^d$  of 1-dimensional subspaces of  $V$  whose sum is  $V$ . For a linear transformation  $A$  from  $V$  to  $V$ , we say  $A$  lowers  $\{V_i\}_{i=0}^d$  whenever  $AV_i = V_{i-1}$  for  $0 \leq i \leq d$ , where  $V_{-1} = 0$ . We say  $A$  raises  $\{V_i\}_{i=0}^d$  whenever  $AV_i = V_{i+1}$  for  $0 \leq i \leq d$ , where  $V_{d+1} = 0$ . An ordered pair of linear transformations  $A, B$  from  $V$  to  $V$  is called LR whenever there exists a decomposition  $\{V_i\}_{i=0}^d$  of  $V$  that is lowered by  $A$  and raised by  $B$ . In this case the decomposition  $\{V_i\}_{i=0}^d$  is uniquely determined by  $A, B$ ; we call it the  $(A, B)$ -decomposition of  $V$ . Consider a 3-tuple of linear transformations  $A, B, C$  from  $V$  to  $V$  such that any two of  $A, B, C$  form an LR pair on  $V$ . Such a 3-tuple is called an LR triple on  $V$ . Let  $\alpha, \beta, \gamma$  be nonzero scalars in  $\mathbb{F}$ . The triple  $\alpha A, \beta B, \gamma C$  is an LR triple on  $V$ , said to be associated to  $A, B, C$ . Let  $\{V_i\}_{i=0}^d$  be a decomposition of  $V$  and let  $X$  be a linear transformation from  $V$  to  $V$ . We say  $X$  is tridiagonal with respect to  $\{V_i\}_{i=0}^d$  whenever  $XV_i \subseteq V_{i-1} + V_i + V_{i+1}$  for  $0 \leq i \leq d$ . Let  $\mathcal{X}$  be the vector space over  $\mathbb{F}$  consisting of the linear transformations from  $V$  to  $V$  that are tridiagonal with respect to the  $(A, B)$  and  $(B, C)$  and  $(C, A)$  decompositions of  $V$ . There is a special class of LR triples, called  $q$ -Weyl type. In the present paper, we find a basis of  $\mathcal{X}$  for each LR triple that is not associated to an LR triple of  $q$ -Weyl type.

## 1 Introduction

The equitable presentation for the quantum algebra  $U_q(\mathfrak{sl}_2)$  was introduced in [3] and further investigated in [4, 5]. For the lie algebra  $\mathfrak{sl}_2$ , the equitable presentation was introduced in [2] and comprehensively studied in [1]. From the equitable point of view, consider a finite-dimensional irreducible module for  $U_q(\mathfrak{sl}_2)$  or  $\mathfrak{sl}_2$ . In [1, 4] three nilpotent linear transformations of the module are encountered, with each transformation acting as a lowering map and raising map in multiple ways. In order to describe this situation more precisely, Paul Terwilliger introduced the notion of a lowering-raising (or LR) triple of linear transformations, and gave their complete classification (see [6]).

There are three decompositions associated with an LR triple. In the present paper, we investigate the linear transformations that act in a tridiagonal manner on each of these three decompositions. In this section, we first recall the notion of an LR triple, and then state our main results.

Throughout the paper, fix an integer  $d \geq 0$ , a field  $\mathbb{F}$ , and a vector space  $V$  over  $\mathbb{F}$  with dimension  $d+1$ . Let  $\text{End}(V)$  denote the  $\mathbb{F}$ -algebra consisting of the  $\mathbb{F}$ -linear transformations from  $V$  to  $V$ , and let  $\text{Mat}_{d+1}(\mathbb{F})$  denote the  $\mathbb{F}$ -algebra consisting of the  $(d+1) \times (d+1)$  matrices that have all entries in  $\mathbb{F}$ . We index the rows and columns by  $0, 1, \dots, d$ .

By a *decomposition* of  $V$  we mean a sequence  $\{V_i\}_{i=0}^d$  of 1-dimensional subspaces of  $V$  such that  $V = \sum_{i=0}^d V_i$  (direct sum). Let  $\{V_i\}_{i=0}^d$  be a decomposition of  $V$ . For notational convenience define  $V_{-1} = 0$  and  $V_{d+1} = 0$ . For  $A \in \text{End}(V)$ , we say  $A$  *lowers*  $\{V_i\}_{i=0}^d$  whenever  $AV_i = V_{i-1}$  for  $0 \leq i \leq d$ . We say  $A$  *raises*  $\{V_i\}_{i=0}^d$  whenever  $AV_i = V_{i+1}$  for  $0 \leq i \leq d$ . An ordered pair  $A, B$  of elements in  $\text{End}(V)$  is called *LR* whenever there exists a decomposition of  $V$  that is lowered by  $A$  and raised by  $B$ . In this case the decomposition  $\{V_i\}_{i=0}^d$  is uniquely determined by  $A, B$  (see [6, Section 3]); we call it the  $(A, B)$ -*decomposition* of  $V$ . For  $0 \leq i \leq d$  define  $E_i \in \text{End}(V)$  such that  $(E_i - I)V_i = 0$  and  $E_i V_j = 0$  for  $0 \leq j \leq d$ ,  $j \neq i$ , where  $I$  denotes the identity in  $\text{End}(V)$ . We have  $E_i E_j = \delta_{i,j} E_i$  for  $0 \leq i, j \leq d$  and  $I = \sum_{i=0}^d E_i$ . We call  $\{E_i\}_{i=0}^d$  the *idempotent sequence* for  $A, B$  (or  $\{V_i\}_{i=0}^d$ ).

A 3-tuple  $A, B, C$  of elements in  $\text{End}(V)$  is called an *LR triple* whenever any two of  $A, B, C$  form an LR pair on  $V$ . We say  $A, B, C$  is *over*  $\mathbb{F}$ . We call  $d$  the *diameter* of  $A, B, C$ .

Let  $A, B, C$  be an LR triple on  $V$  and let  $A', B', C'$  be an LR triple on a vector space  $V'$  over  $\mathbb{F}$  with dimension  $d+1$ . By an *isomorphism of LR triples* from  $A, B, C$  to  $A', B', C'$  we mean an  $\mathbb{F}$ -linear bijection  $\sigma : V \rightarrow V'$  such that  $\sigma A = A' \sigma$ ,  $\sigma B = B' \sigma$ ,  $\sigma C = C' \sigma$ . The LR triples  $A, B, C$  and  $A', B', C'$  are said to be *isomorphic* whenever there exists an isomorphism of LR triples from  $A, B, C$  to  $A', B', C'$ .

Let  $A, B, C$  be an LR triple on  $V$ . Let  $\{V_i\}_{i=0}^d$  (resp.  $\{V'_i\}_{i=0}^d$ ) (resp.  $\{V''_i\}_{i=0}^d$ ) be the  $(A, B)$ -decomposition (resp.  $(B, C)$ -decomposition) (resp.  $(B, C)$ -decomposition) of  $V$ , and let  $\{E_i\}_{i=0}^d$  (resp.  $\{E'_i\}_{i=0}^d$ ) (resp.  $\{E''_i\}_{i=0}^d$ ) be the corresponding idempotent sequence. We call the sequence

$$(\{E_i\}_{i=0}^d, \{E'_i\}_{i=0}^d, \{E''_i\}_{i=0}^d) \quad (1)$$

the *idempotent data* of  $A, B, C$ . Define scalars

$$a_i = \text{tr}(CE_i), \quad a'_i = \text{tr}(AE'_i), \quad a''_i = \text{tr}(BE''_i) \quad (0 \leq i \leq d),$$

where  $\text{tr}$  means trace. We call the sequence

$$(\{a_i\}_{i=0}^d, \{a'_i\}_{i=0}^d, \{a''_i\}_{i=0}^d) \quad (2)$$

the *trace data* of  $A, B, C$ . The LR triple is said to be *bipartite* whenever each of  $a_i, a'_i, a''_i$  is zero for  $0 \leq i \leq d$ . In this case,  $d$  is even (see [6, Lemma 16.6]); set  $d = 2m$ . The elements  $\sum_{j=0}^m E_{2j}$ ,  $\sum_{j=0}^m E'_{2j}$ ,  $\sum_{j=0}^m E''_{2j}$  are equal (see [6, Lemma 16.12]). We denote this element by  $J$ :

$$J = \sum_{j=0}^m E_{2j} = \sum_{j=0}^m E'_{2j} = \sum_{j=0}^m E''_{2j}. \quad (3)$$

Observe

$$I - J = \sum_{j=0}^{m-1} E_{2j+1} = \sum_{j=0}^{m-1} E'_{2j+1} = \sum_{j=0}^{m-1} E''_{2j+1}. \quad (4)$$

Let  $A, B, C$  be an LR triple on  $V$ , and  $\alpha, \beta, \gamma$  be nonzero scalars in  $\mathbb{F}$ . Then  $\alpha A, \beta B, \gamma C$  is an LR triple on  $V$ , and this LR triple has the same idempotent data as  $A, B, C$  (see [6, Lemma 13.22]). Two LR triples  $A, B, C$  and  $A', B', C'$  over  $\mathbb{F}$  are said to be *associated* whenever there exist nonzero scalars  $\alpha, \beta, \gamma$  in  $\mathbb{F}$  such that  $A' = \alpha A$ ,  $B' = \beta B$ ,  $C' = \gamma C$ .

There is a special class of LR triples, said to have  $q$ -Weyl type. This is described as follows. Let  $0 \neq q \in \mathbb{F}$ . An LR pair  $A, B$  on  $V$  is said to have  $q$ -Weyl type whenever  $q^2 \neq 1$  and

$$\frac{qAB - q^{-1}BA}{q - q^{-1}} = I.$$

In this case,  $d \geq 1$  and  $q$  is a  $(2d + 2)$ -root of unity (see [6, Lemma 4.16]). An LR triple  $A, B, C$  on  $V$  is said to have  $q$ -Weyl type whenever the LR pairs  $A, B$  and  $B, C$  and  $C, A$  all have  $q$ -Weyl type.

Let  $X \in \text{End}(V)$  and let  $\{V_i\}_{i=0}^d$  be a decomposition of  $V$ . We say  $X$  is *tridiagonal with respect to*  $\{V_i\}_{i=0}^d$  whenever

$$XV_i \subseteq V_{i-1} + V_i + V_{i+1} \quad (0 \leq i \leq d). \quad (5)$$

Let  $\{E_i\}_{i=0}^d$  be the idempotent sequence for  $\{V_i\}_{i=0}^d$ . Then  $X$  satisfies (5) if and only if  $E_r X E_s = 0$  if  $|r - s| > 1$  ( $0 \leq r, s \leq d$ ).

Let  $A, B, C$  be an LR triple on  $V$ . Let  $\mathcal{X}$  denote the subspace of  $\text{End}(V)$  consisting of  $X \in \text{End}(V)$  such that  $X$  is tridiagonal with respect to the  $(A, B)$  and  $(B, C)$  and  $(C, A)$  decompositions for  $V$ . We call  $\mathcal{X}$  the *tridiagonal space* for  $A, B, C$ . Each LR triple on  $V$  that is associated to  $A, B, C$  has tridiagonal space  $\mathcal{X}$  (see Corollary 5.2). The following elements are contained in  $\mathcal{X}$  (see Lemma 5.4):

$$I, \quad A, \quad B, \quad C, \quad ABC, \quad BCA, \quad CAB, \quad ACB, \quad CBA, \quad BAC. \quad (6)$$

Moreover, if  $A, B, C$  is bipartite, then  $XJ \in \mathcal{X}$  for any  $X \in \mathcal{X}$  (see Lemma 11.4). In the present paper, we investigate the tridiagonal space  $\mathcal{X}$ . Observe that  $\mathcal{X} = \text{End}(V)$  when  $d \leq 1$ . So we restrict our attention to the case  $d \geq 2$ . We prove the following results:

**Theorem 1.1** *Let  $A, B, C$  be a nonbipartite LR triple on  $V$ , and let  $\mathcal{X}$  be the corresponding tridiagonal space. Assume that  $A, B, C$  is not associated to an LR triple of  $q$ -Weyl type. Then the following hold:*

- (i) *Assume  $d = 2$ . Then  $\mathcal{X}$  has dimension 6. Moreover, the vector space  $\mathcal{X}$  has a basis*

$$I, \quad A, \quad B, \quad C, \quad ABC, \quad ACB. \quad (7)$$

- (ii) *Assume  $d \geq 3$ . Then  $\mathcal{X}$  has dimension 7. Moreover, the vector space  $\mathcal{X}$  has a basis*

$$I, \quad A, \quad B, \quad C, \quad ABC, \quad ACB, \quad CAB. \quad (8)$$

**Theorem 1.2** *Let  $A, B, C$  be a bipartite LR triple on  $V$ , and let  $\mathcal{X}$  be the corresponding tridiagonal space. Then  $\mathcal{X} = \mathcal{X}J + \mathcal{X}(I - J)$  (direct sum). Moreover, the following hold:*

(i) *Assume  $d = 2$ . Then  $\mathcal{X}$  has dimension 6. The vector space  $\mathcal{X}J$  has a basis*

$$J, \quad AJ, \quad BJ, \tag{9}$$

*and the space  $\mathcal{X}(I - J)$  has a basis*

$$I - J, \quad A(I - J), \quad B(I - J). \tag{10}$$

(ii) *Assume  $d \geq 4$ . Then  $\mathcal{X}$  has dimension 8. The vector space  $\mathcal{X}J$  has a basis*

$$J, \quad AJ, \quad BJ, \quad ACBJ, \tag{11}$$

*and the space  $\mathcal{X}(I - J)$  has a basis*

$$I - J, \quad A(I - J), \quad B(I - J), \quad ABC(I - J). \tag{12}$$

**Note 1.3** When  $A, B, C$  has  $q$ -Weyl type, the elements

$$ABC, \quad BCA, \quad CAB, \quad ACB, \quad CBA, \quad BAC$$

are contained in the span of  $I, A, B, C$  (see [6, Lemma 15.30]).

**Problem 1.4** For an LR triple  $A, B, C$  of  $q$ -Weyl type, find the dimension and a basis for the tridiagonal space  $\mathcal{X}$ .

The paper is organized as follows. In Section 2 we consider 12 bases for  $V$ . In Section 3 we obtain the transition matrices between these 12 bases. In Section 4 we obtain the matrices that represent the idempotents with respect to the 12 bases. In Section 5 we prepare some lemmas concerning the tridiagonal space. In Sections 6–8 we consider nonbipartite LR triples. In Section 6 we recall the classification of nonbipartite LR triples. In Sections 7 and 8 we prove Theorem 1.1. In Sections 9–11 we consider bipartite LR triples. In Section 9 we recall the classification of bipartite LR triples. In Sections 10 and 11 we prove Theorem 1.2. In Appendix 1 we represent the elements (6) in terms of (8). In Appendix 2 we represent the elements (6) times  $J$  in terms of (11), and represent the elements (6) times  $I - J$  in terms of (12).

## 2 Some bases for $V$

Let  $A, B$  be an LR pair on  $V$ , and let  $\{V_i\}_{i=0}^d$  be the  $(A, B)$ -decomposition of  $V$ . By [6, Lemma 3.12], for  $0 \leq i \leq d$  the subspace  $V_i$  is invariant under  $AB$  and  $BA$ . Moreover, for  $1 \leq i \leq d$ , the eigenvalue of  $AB$  on  $V_{i-1}$  is nonzero and equal to the eigenvalue of  $BA$  on  $V_i$ . We denote this eigenvalue by  $\varphi_i$ . The sequence  $\{\varphi_i\}_{i=1}^d$  is called the *parameter sequence* for  $A, B$ . We emphasize that  $\varphi_i \neq 0$  for  $1 \leq i \leq d$ . For notational convenience define

$\varphi_0 = 0$  and  $\varphi_{d+1} = 0$ . A basis  $\{v_i\}_{i=0}^d$  for  $V$  is called an  $(A, B)$ -basis whenever  $v_i \in V_i$  for  $0 \leq i \leq d$  and  $Av_i = v_{i-1}$  for  $1 \leq i \leq d$ . A basis  $\{v_i\}_{i=0}^d$  for  $V$  is called an *inverted*  $(A, B)$ -basis whenever its inversion  $\{v_{d-i}\}_{i=0}^d$  is an  $(A, B)$ -basis for  $V$ .

Let  $A, B, C$  be an LR triple on  $V$ . As we discuss this LR triple, we will use the following notation:

**Definition 2.1** (See [6, Definition 13.4].) Let  $A, B, C$  be an LR triple. For any object  $f$  associated with the LR triple  $A, B, C$ , let  $f'$  (resp.  $f''$ ) denote the corresponding object for the LR triple  $B, C, A$  (resp.  $C, A, B$ ).

**Definition 2.2** (See [6, Definition 13.21].) Let  $A, B, C$  be an LR triple on  $V$ . So the pair  $A, B$  (resp.  $B, C$ ) (resp.  $C, A$ ) is an LR pair on  $V$ . Following the notational convention in Definition 2.1, for these LR pairs the parameter sequence is denoted as follows:

LR pair	parameter sequence
$A, B$	$\{\varphi_i\}_{i=1}^d$
$B, C$	$\{\varphi'_i\}_{i=1}^d$
$C, A$	$\{\varphi''_i\}_{i=1}^d$

We call the sequence

$$(\{\varphi_i\}_{i=1}^d, \{\varphi'_i\}_{i=1}^d, \{\varphi''_i\}_{i=1}^d) \quad (13)$$

the *parameter array* of the LR triple  $A, B, C$ .

Let  $A, B, C$  be an LR triple on  $V$ . Associated with  $A, B, C$  are 12 types of bases for  $V$ :

$$\begin{array}{llll} (A, B), & \text{inverted } (A, B), & (B, A), & \text{inverted } (B, A), \\ (B, C), & \text{inverted } (B, C), & (C, B), & \text{inverted } (C, B), \\ (C, A), & \text{inverted } (C, A), & (A, C), & \text{inverted } (A, C). \end{array} \quad (14)$$

**Lemma 2.3** (See [6, Lemma 13.19].) Let  $\{v_i\}_{i=0}^d$  be a basis for  $V$  that has one of the 12 types (14). Then for each  $A, B, C$ , the matrix representing it with respect to  $\{v_i\}_{i=0}^d$  is tridiagonal.

Let (13) be the parameter array and let (2) be the trace data of  $A, B, C$ .

**Lemma 2.4** (See [6, Proposition 13.39].) Fix an  $(A, B)$ -basis  $\{v_i\}_{i=0}^d$  for  $V$ . Identify each element of  $\text{End}(V)$  with the matrix representing it with respect to  $\{v_i\}_{i=0}^d$ . Then each of  $A, B, C$  is tridiagonal with the following entries:

$A_{i,i-1}$	$A_{i,i}$	$A_{i-1,i}$	$B_{i,i-1}$	$B_{i,i}$	$B_{i-1,i}$	$C_{i,i-1}$	$C_{i,i}$	$C_{i-1,i}$
0	0	1	$\varphi_i$	0	0	$\varphi''_{d-i+1}$	$a_i$	$\varphi'_{d-i+1}/\varphi_i$

### 3 Transition matrices

Let  $A, B, C$  be an LR triple on  $V$  with parameter array (13). In this section, we consider the transition matrices between the 12 bases (14).

**Definition 3.1** Two bases  $\{v_i\}_{i=0}^d$  and  $\{u_i\}_{i=0}^d$  for  $V$  are said to be *compatible* whenever  $v_0 = u_0$ .

**Definition 3.2** (See [6, Definition 13.44].) Define matrices  $T, T', T''$  in  $\text{Mat}_{d+1}(\mathbb{F})$  as follows:

$T$  is the transition matrix from a  $(C, B)$ -basis to a compatible  $(C, A)$ -basis;

$T'$  is the transition matrix from an  $(A, C)$ -basis to a compatible  $(A, B)$ -basis;

$T''$  is the transition matrix from a  $(B, A)$ -basis to a compatible  $(B, C)$ -basis.

Let  $\{\alpha_i\}_{i=0}^d$  be scalars in  $\mathbb{F}$ . An upper triangular matrix  $T \in \text{Mat}_{d+1}(\mathbb{F})$  is called *Toeplitz with parameters*  $\{\alpha_i\}_{i=0}^d$  whenever  $T$  has  $(i, j)$ -entry  $\alpha_{j-i}$  for  $0 \leq i \leq j \leq d$ :

$$T = \begin{pmatrix} \alpha_0 & \alpha_1 & \cdot & \cdot & \cdot & \alpha_d \\ & \alpha_0 & \alpha_1 & \cdot & \cdot & \cdot \\ & & \alpha_0 & \cdot & \cdot & \cdot \\ & & & \cdot & \cdot & \cdot \\ & & & & \cdot & \alpha_1 \\ \mathbf{0} & & & & & \alpha_0 \end{pmatrix}.$$

This matrix is invertible if and only if  $\alpha_0 \neq 0$ . In this case,  $T^{-1}$  is upper triangular and Toeplitz (see [6, Section 12]).

**Lemma 3.3** (See [6, Proposition 12.8].) *With reference to Definition 3.2, each of  $T, T', T''$  is upper triangular and Toeplitz.*

**Definition 3.4** (See [6, Definition 13.45].) With reference to Lemma 3.3, let  $\{\alpha_i\}_{i=0}^d$  (resp.  $\{\alpha'_i\}_{i=0}^d$ ) (resp.  $\{\alpha''_i\}_{i=0}^d$ ) be the Toeplitz parameters for  $T$  (resp.  $T'$ ) (resp.  $T''$ ). Let  $\{\beta_i\}_{i=0}^d$  (resp.  $\{\beta'_i\}_{i=0}^d$ ) (resp.  $\{\beta''_i\}_{i=0}^d$ ) be the Toeplitz parameters for  $T^{-1}$  (resp.  $(T')^{-1}$ ) (resp.  $(T'')^{-1}$ ). We call the sequence

$$(\{\alpha_i\}_{i=0}^d, \{\beta_i\}_{i=0}^d; \{\alpha'_i\}_{i=0}^d, \{\beta'_i\}_{i=0}^d; \{\alpha''_i\}_{i=0}^d, \{\beta''_i\}_{i=0}^d) \quad (15)$$

the *Toeplitz data* for  $A, B, C$ .

**Lemma 3.5** (See [6, Lemma 13.46].) *With reference to Definition 3.4,*

$$\alpha_0 = 1, \quad \alpha'_0 = 1, \quad \alpha''_0 = 1, \quad \beta_0 = 1, \quad \beta'_0 = 1, \quad \beta''_0 = 1.$$

Moreover, when  $d \geq 1$ ,

$$\beta_1 = -\alpha_1, \quad \beta'_1 = -\alpha'_1, \quad \beta''_1 = -\alpha''_1.$$

**Definition 3.6** Let  $Z$  denote the matrix in  $\text{Mat}_{d+1}(\mathbb{F})$  that has  $(i, j)$ -entry  $\delta_{i+j, d}$  for  $0 \leq i, j \leq d$ . For example if  $d = 3$ ,

$$Z = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

Observe that  $Z$  is invertible and  $Z^{-1} = Z$ . Let  $D$  (resp.  $D'$ ) (resp.  $D''$ ) denote the diagonal matrix in  $\text{Mat}_{d+1}(\mathbb{F})$  that has  $(i, i)$ -entry  $\varphi_1 \cdots \varphi_i$  (resp.  $\varphi'_1 \cdots \varphi'_i$ ) (resp.  $\varphi''_1 \cdots \varphi''_i$ ) for  $0 \leq i \leq d$ .

**Lemma 3.7** (See [6, Lemma 3.48].)

- (i) *The transition matrix from an  $(A, B)$ -basis to an inverted  $(A, B)$ -basis is a nonzero scalar multiple of  $Z$ .*
- (ii) *The transition matrix from an  $(A, B)$ -basis to an inverted  $(B, A)$ -basis is a nonzero scalar multiple of  $D$ .*
- (iii) *The transition matrix from an  $(A, B)$ -basis to a  $(B, A)$ -basis is a nonzero scalar multiple of  $DZ$ .*

**Lemma 3.8** *In the table below, the transition matrix from the basis in the first column to the basis in the second column is a nonzero scalar multiple of the matrix in the third column:*

from	to	transition matrix
$(A, B)$	$(A, B)$	$I$
$(A, B)$	inv. $(A, B)$	$Z$
$(A, B)$	$(B, A)$	$DZ$
$(A, B)$	inv. $(B, A)$	$D$
$(A, C)$	$(A, B)$	$T'$
$(A, C)$	inv. $(A, B)$	$T'Z$
$(A, C)$	$(B, A)$	$T'DZ$
$(A, C)$	inv. $(B, A)$	$T'D$
$(B, C)$	$(A, B)$	$(T'')^{-1}ZD^{-1}$
$(B, C)$	inv. $(A, B)$	$(T'')^{-1}ZD^{-1}Z$
$(B, C)$	$(B, A)$	$(T'')^{-1}$
$(B, C)$	inv. $(B, A)$	$(T'')^{-1}Z$

**Proof.** The transition matrices from an  $(A, B)$ -basis are given in Lemma 3.7. By Definition 3.2 the transition matrix from an  $(A, C)$ -basis to an  $(A, B)$ -basis is a nonzero scalar multiple of  $T'$ . The remaining transition matrices from an  $(A, C)$ -basis are obtained by multiplying  $T'$  on the right by the transition matrices from an  $(A, B)$ -basis. By Definition 3.2 the transition matrix from an  $(B, A)$ -basis to a  $(B, C)$ -basis is a nonzero scalar multiple of  $T''$ , so the transition matrix from a  $(B, C)$ -basis to a  $(B, A)$ -basis is a nonzero scalar multiple

of  $(T'')^{-1}$ . The transition matrix from an  $(A, B)$ -basis to a  $(B, A)$ -basis is a nonzero scalar multiple of  $DZ$ , so the transition matrix from a  $(B, A)$ -basis to an  $(A, B)$ -basis is a nonzero scalar multiple of  $(DZ)^{-1}$ . By these comments, the transition matrix from a  $(B, C)$ -basis to an  $(A, B)$ -basis is a nonzero scalar multiple of  $(T'')^{-1}(DZ)^{-1} = (T'')^{-1}ZD^{-1}$ . The remaining transition matrices from a  $(B, C)$ -basis are obtained by multiplying  $(T'')^{-1}ZD^{-1}$  on the right by the transition matrices from an  $(A, B)$ -basis.  $\square$

Applying Lemma 3.8 to the LR triple  $B, C, A$  we obtain:

**Lemma 3.9** *In the table below, the transition matrix from the basis in the first column to the basis in the second column is a nonzero scalar multiple of the matrix in the third column:*

from	to	transition matrix
$(B, C)$	$(B, C)$	$I$
$(B, C)$	$\text{inv.}(B, C)$	$Z$
$(B, C)$	$(C, B)$	$D'Z$
$(B, C)$	$\text{inv.}(C, B)$	$D'$
$(B, A)$	$(B, C)$	$T''$
$(B, A)$	$\text{inv.}(B, C)$	$T''Z$
$(B, A)$	$(C, B)$	$T''D'Z$
$(B, A)$	$\text{inv.}(C, B)$	$T''D'$
$(C, A)$	$(B, C)$	$T^{-1}Z(D')^{-1}$
$(C, A)$	$\text{inv.}(B, C)$	$T^{-1}Z(D')^{-1}Z$
$(C, A)$	$(C, B)$	$T^{-1}$
$(C, A)$	$\text{inv.}(C, B)$	$T^{-1}Z$

Applying Lemma 3.8 to the LR triple  $C, A, B$  we obtain:

**Lemma 3.10** *In the table below, the transition matrix from the basis in the first column to the basis in the second column is a nonzero scalar multiple of the matrix in the third column:*

from	to	transition matrix
$(C, A)$	$(C, A)$	$I$
$(C, A)$	$\text{inv.}(C, A)$	$Z$
$(C, A)$	$(A, C)$	$D''Z$
$(C, A)$	$\text{inv.}(A, C)$	$D''$
$(C, B)$	$(C, A)$	$T$
$(C, B)$	$\text{inv.}(C, A)$	$TZ$
$(C, B)$	$(A, C)$	$TD''Z$
$(C, B)$	$\text{inv.}(A, C)$	$TD''$
$(A, B)$	$(C, A)$	$(T')^{-1}Z(D'')^{-1}$
$(A, B)$	$\text{inv.}(C, A)$	$(T')^{-1}Z(D'')^{-1}Z$
$(A, B)$	$(A, C)$	$(T')^{-1}$
$(A, B)$	$\text{inv.}(A, C)$	$(T')^{-1}Z$



## 4 Representing the idempotents with respect to the 12 bases

In this section, we obtain the matrices that represent the idempotents with respect to the 12 bases (14). We begin by recalling a lemma from elementary linear algebra:

**Lemma 4.1** *Let  $H \in \text{End}(V)$ , and  $\{u_i\}_{i=0}^d, \{v_i\}_{i=0}^d$  be bases for  $V$ . Let  $M$  be the matrix representing  $H$  with respect to  $\{u_i\}_{i=0}^d$ , and let  $S$  be the transition matrix from  $\{u_i\}_{i=0}^d$  to  $\{v_i\}_{i=0}^d$ . Then the matrix representing  $H$  with respect to  $\{v_i\}_{i=0}^d$  is  $S^{-1}MS$ .*

We use the following notation:

**Definition 4.2** For  $0 \leq r \leq d$  let  $F_r$  denote the matrix in  $\text{Mat}_{d+1}(\mathbb{F})$  that has  $(r, r)$ -entry 1 and all other entries 0.

Let  $A, B, C$  be an LR triple on  $V$  with parameter array (13), Toeplitz data (15), and idempotent data (1). The following three propositions are routinely obtained from Lemmas 3.8–3.10 and 4.1.

**Proposition 4.3** *For  $0 \leq r \leq d$ , with respect to the basis in the first column,  $E_r$  is represented by the matrix in the second column:*

basis	the matrix representing $E_r$
$(A, B)$	$F_r$
$\text{inv.}(A, B)$	$F_{d-r}$
$(B, A)$	$F_{d-r}$
$\text{inv.}(B, A)$	$F_r$
$(B, C)$	$(T'')^{-1}F_{d-r}T''$
$\text{inv.}(B, C)$	$Z(T'')^{-1}F_{d-r}T''Z$
$(C, B)$	$Z(D')^{-1}(T'')^{-1}F_{d-r}T''D'Z$
$\text{inv.}(C, B)$	$(D')^{-1}(T'')^{-1}F_{d-r}T''D'$
$(C, A)$	$D''ZT'F_r(T')^{-1}Z(D'')^{-1}$
$\text{inv.}(C, A)$	$ZD''ZT'F_r(T')^{-1}Z(D'')^{-1}Z$
$(A, C)$	$T'F_r(T')^{-1}$
$\text{inv.}(A, C)$	$ZT'F_r(T')^{-1}Z$

**Proposition 4.4** For  $0 \leq r \leq d$ , with respect to the basis in the first column,  $E'_r$  is represented by the matrix in the second column:

basis	the matrix representing $E'_r$
$(A, B)$	$DZT''F_r(T'')^{-1}ZD^{-1}$
$\text{inv.}(A, B)$	$ZDZT''F_r(T'')^{-1}ZD^{-1}Z$
$(B, A)$	$T''F_r(T'')^{-1}$
$\text{inv.}(B, A)$	$ZT''F_r(T'')^{-1}Z$
$(B, C)$	$F_r$
$\text{inv.}(B, C)$	$F_{d-r}$
$(C, B)$	$F_{d-r}$
$\text{inv.}(C, B)$	$F_r$
$(C, A)$	$T^{-1}F_{d-r}T$
$\text{inv.}(C, A)$	$ZT^{-1}F_{d-r}TZ$
$(A, C)$	$Z(D'')^{-1}T^{-1}F_{d-r}TD''Z$
$\text{inv.}(A, C)$	$(D'')^{-1}T^{-1}F_{d-r}TD''$

**Proposition 4.5** For  $0 \leq r \leq d$ , with respect to the basis in the first column,  $E''_r$  is represented by the matrix in the second column:

basis	the matrix representing $E''_r$
$(A, B)$	$(T')^{-1}F_{d-r}T'$
$\text{inv.}(A, B)$	$Z(T')^{-1}F_{d-r}T'Z$
$(B, A)$	$ZD^{-1}(T')^{-1}F_{d-r}T'DZ$
$\text{inv.}(B, A)$	$D^{-1}(T')^{-1}F_{d-r}T'D$
$(B, C)$	$D'ZTF_rT^{-1}Z(D')^{-1}$
$\text{inv.}(B, C)$	$ZD'ZTF_rT^{-1}Z(D')^{-1}Z$
$(C, B)$	$TF_rT^{-1}$
$\text{inv.}(C, B)$	$ZTF_rT^{-1}Z$
$(C, A)$	$F_r$
$\text{inv.}(C, A)$	$F_{d-r}$
$(A, C)$	$F_{d-r}$
$\text{inv.}(A, C)$	$F_r$

The following three proposition are obtained by computing the entries of the matrices given in Propositions 4.3–4.5.

**Proposition 4.6** For  $0 \leq r, i, j \leq d$ , with respect to the basis in the first column, the matrix representing  $E_r$  has  $(i, j)$ -entry in the second column when the condition in the third column is satisfied, and 0 otherwise:

basis	$(i, j)$ -entry of the matrix representing $E_r$	condition
$(A, B)$	1	$i = r = j$
$\text{inv.}(A, B)$	1	$i = d - r = j$
$(B, A)$	1	$i = d - r = j$
$\text{inv.}(B, A)$	1	$i = r = j$
$(B, C)$	$\alpha''_{r-d+j}\beta''_{d-r-i}$	$i \leq d - r \leq j$
$\text{inv.}(B, C)$	$\alpha''_{r-j}\beta''_{i-r}$	$j \leq r \leq i$
$(C, B)$	$\alpha''_{r-j}\beta''_{i-r}\varphi'_{d-i+1} \cdots \varphi'_{d-j}$	$j \leq r \leq i$
$\text{inv.}(C, B)$	$\alpha''_{r-d+j}\beta''_{d-r-i}\varphi'_{i+1} \cdots \varphi'_j$	$i \leq d - r \leq j$
$(C, A)$	$\alpha'_{r-d+i}\beta'_{d-r-j}\varphi''_{j+1} \cdots \varphi''_i$	$j \leq d - r \leq i$
$\text{inv.}(C, A)$	$\alpha'_{r-i}\beta'_{j-r}\varphi''_{d-j+1} \cdots \varphi''_{d-i}$	$i \leq r \leq j$
$(A, C)$	$\alpha'_{r-i}\beta'_{j-r}$	$i \leq r \leq j$
$\text{inv.}(A, C)$	$\alpha'_{r-d+i}\beta'_{d-r-j}$	$j \leq d - r \leq i$

**Proposition 4.7** For  $0 \leq r, i, j \leq d$ , with respect to the basis in the first column, the matrix representing  $E'_r$  has  $(i, j)$ -entry in the second column when the condition in the third column is satisfied, and 0 otherwise:

basis	$(i, j)$ -entry of the matrix representing $E'_r$	condition
$(A, B)$	$\alpha''_{r-d+i}\beta''_{d-r-j}\varphi_{j+1} \cdots \varphi_i$	$j \leq d - r \leq i$
$\text{inv.}(A, B)$	$\alpha''_{r-i}\beta''_{j-r}\varphi_{d-j+1} \cdots \varphi_{d-i}$	$i \leq r \leq j$
$(B, A)$	$\alpha''_{r-i}\beta''_{j-r}$	$i \leq r \leq j$
$\text{inv.}(B, A)$	$\alpha''_{r-d+i}\beta''_{d-r-j}$	$j \leq d - r \leq i$
$(B, C)$	1	$i = r = j$
$\text{inv.}(B, C)$	1	$i = d - r = j$
$(C, B)$	1	$i = d - r = j$
$\text{inv.}(C, B)$	1	$i = r = j$
$(C, A)$	$\alpha_{r-d+j}\beta_{d-r-i}$	$i \leq d - r \leq j$
$\text{inv.}(C, A)$	$\alpha_{r-j}\beta_{i-r}$	$j \leq r \leq i$
$(A, C)$	$\alpha_{r-j}\beta_{i-r}\varphi''_{d-i+1} \cdots \varphi''_{d-j}$	$j \leq r \leq i$
$\text{inv.}(A, C)$	$\alpha_{r-d+j}\beta_{d-r-i}\varphi''_{i+1} \cdots \varphi''_j$	$i \leq d - r \leq j$

**Proposition 4.8** For  $0 \leq r, i, j \leq d$ , with respect to the basis in the first column, the matrix representing  $E_r''$  has  $(i, j)$ -entry in the second column when the condition in the third column is satisfied, and 0 otherwise:

basis	$(i, j)$ -entry of the matrix representing $E_r''$	condition
$(A, B)$	$\alpha'_{r-d+j}\beta'_{d-r-i}$	$i \leq d-r \leq j$
$\text{inv.}(A, B)$	$\alpha'_{r-j}\beta'_{i-r}$	$j \leq r \leq i$
$(B, A)$	$\alpha'_{r-j}\beta'_{i-r}\varphi_{d-i+1} \cdots \varphi_{d-j}$	$j \leq r \leq i$
$\text{inv.}(B, A)$	$\alpha'_{r-d+j}\beta'_{d-r-i}\varphi_{i+1} \cdots \varphi_j$	$i \leq d-r \leq j$
$(B, C)$	$\alpha_{r-d+i}\beta_{d-r-j}\varphi'_{j+1} \cdots \varphi'_i$	$j \leq d-r \leq i$
$\text{inv.}(B, C)$	$\alpha_{r-i}\beta_{j-r}\varphi'_{d-j+1} \cdots \varphi'_{d-i}$	$i \leq r \leq j$
$(C, B)$	$\alpha_{r-i}\beta_{j-r}$	$i \leq r \leq j$
$\text{inv.}(C, B)$	$\alpha_{r-d+i}\beta_{d-r-j}$	$j \leq d-r \leq i$
$(C, A)$	1	$i = r = j$
$\text{inv.}(C, A)$	1	$i = d-r = j$
$(A, C)$	1	$i = d-r = j$
$\text{inv.}(A, C)$	1	$i = r = j$

## 5 Some lemmas concerning the tridiagonal space

In this section we prepare some lemmas concerning the tridiagonal space that we need in our proof of Theorems 1.1 and 1.2. Let  $A, B, C$  be an LR triple on  $V$ .

**Lemma 5.1** (See [6, Lemma 13.22].) *Let  $\alpha, \beta, \gamma$  be nonzero scalars in  $\mathbb{F}$ . Then the 3-tuple  $\alpha A, \beta B, \gamma C$  is an LR triple on  $V$ . Moreover, the idempotent data of this LR triple is equal to the idempotent data of  $A, B, C$ .*

**Corollary 5.2** *With reference to Lemma 5.1, the LR triples  $A, B, C$  and  $\alpha A, \beta B, \gamma C$  have the same tridiagonal space.*

**Proof.** Follows from Lemma 5.1. □

**Lemma 5.3** (See [6, Lemma 13.19].) *In each row of the table below, we display a decomposition  $\{V_i\}_{i=0}^d$  of  $V$ . For  $0 \leq i \leq d$  we give the action of  $A, B, C$  on  $V_i$ .*

dec. $\{V_i\}_{i=0}^d$	action of $A$ on $V_i$	action of $B$ on $V_i$	action of $C$ on $V_i$
$(A, B)$	$AV_i = V_{i-1}$	$BV_i = V_{i+1}$	$CV_i \subseteq V_{i-1} + V_i + V_{i+1}$
$(B, C)$	$AV_i \subseteq V_{i-1} + V_i + V_{i+1}$	$BV_i = V_{i-1}$	$CV_i = V_{i+1}$
$(C, A)$	$AV_i = V_{i+1}$	$BV_i \subseteq V_{i-1} + V_i + V_{i+1}$	$CV_i = V_{i-1}$

**Lemma 5.4** *The elements (6) are contained in the tridiagonal space for  $A, B, C$ .*

**Proof.** Let  $\mathcal{X}$  be the tridiagonal space for  $A, B, C$ . Clearly  $I$  is contained in  $\mathcal{X}$ . By Lemma 5.3 each of  $A, B, C$  is tridiagonal with respect to the  $(A, B)$ -basis,  $(B, C)$ -basis,

and  $(C, A)$ -basis. So  $A, B, C$  are contained in  $\mathcal{X}$ . We show  $ABC$  is contained in  $\mathcal{X}$ . Let  $\{V_i\}_{i=0}^d$  be the  $(A, B)$ -decomposition of  $V$ . Pick any  $i$  ( $0 \leq i \leq d$ ). By Lemma 5.3  $ABV_i \subseteq V_i$ , and  $CV_i \subseteq V_{i-1} + V_i + V_{i+1}$ . By these comments  $ABCV_i \subseteq V_{i-1} + V_i + V_{i+1}$ . So  $ABC$  is tridiagonal with respect to  $\{V_i\}_{i=0}^d$ . In a similar way, we can show that  $ABC$  is tridiagonal with respect to the  $(B, C)$ -decomposition and the  $(C, A)$ -decomposition. We have shown that  $ABC$  is contained in  $\mathcal{X}$ . The proof is similar for the remaining elements in (6).  $\square$

**Lemma 5.5** *Assume  $d = 2$ . Then the tridiagonal space for  $A, B, C$  has dimension at most 6.*

**Proof.** Let  $\mathcal{X}$  be the tridiagonal space for  $A, B, C$ . Fix an  $(A, B)$ -basis  $\{v_i\}_{i=0}^d$  for  $V$ . We identify each element of  $\text{End}(V)$  with the matrix that represents it with respect to  $\{v_i\}_{i=0}^d$ . Pick any  $X \in \mathcal{X}$ . By construction,  $X$  is tridiagonal, so  $X_{0,2} = 0$  and  $X_{2,0} = 0$ . By the definition of  $\mathcal{X}$  we have  $E_2'' X E_0'' = 0$ . Let (15) be the Toeplitz data of  $A, B, C$ . By Proposition 4.8

$$\begin{aligned} (E_2'')_{0,i} &= \alpha'_i \beta'_0 & (0 \leq i \leq d), \\ (E_0'')_{j,2} &= \alpha'_0 \beta'_{2-j} & (0 \leq j \leq d). \end{aligned}$$

Using this we compute the  $(0, 2)$ -entry of  $E_2'' X E_0''$  to find that  $\alpha'_0 \beta'_0$  times

$$\alpha'_0 \beta'_2 X_{0,0} + \alpha'_0 \beta'_1 X_{0,1} + \alpha'_1 \beta'_2 X_{1,0} + \alpha'_1 \beta'_1 X_{1,1} + \alpha'_1 \beta'_0 X_{1,2} + \alpha'_2 \beta'_1 X_{2,1} + \alpha'_2 \beta'_0 X_{2,2} \quad (16)$$

is 0. By Lemma 3.5  $\alpha'_0 = 1$  and  $\beta'_0 = 1$ . So (16) is 0. By [6, Lemma 13.62]  $\beta'_2 \neq 0$ . Therefore  $X_{0,0}$  is uniquely determined by the 6 entries  $X_{0,1}, X_{1,0}, X_{1,1}, X_{1,2}, X_{2,1}, X_{2,2}$ . Therefore  $\mathcal{X}$  has dimension at most 6.  $\square$

## 6 Nonbipartite LR triples

In this section, we recall the classification of nonbipartite LR triples. Let  $A, B, C$  be an LR triple on  $V$  with parameter array (13), Toeplitz data (15), and idempotent data (1). To avoid triviality, we assume  $d \geq 1$ .

**Lemma 6.1** (See [6, Lemma 16.5].) *Assume  $A, B, C$  is nonbipartite. Then each of  $\alpha_1, \alpha'_1, \alpha''_1, \beta_1, \beta'_1, \beta''_1$  is nonzero.*

**Definition 6.2** (See [6, Definitions 18.2].) *Assume  $A, B, C$  is nonbipartite. Then  $A, B, C$  is said to be *normalized* whenever  $\alpha_1 = 1, \alpha'_1 = 1, \alpha'' = 1$ .*

**Lemma 6.3** (See [6, Corollary 18.6].) *Assume  $A, B, C$  is nonbipartite. Then there exist nonzero scalars  $\alpha, \beta, \gamma$  in  $\mathbb{F}$  such that  $\alpha A, \beta B, \gamma C$  is normalized.*

**Lemma 6.4** (See [6, Lemmas 17.6, 18.3].) *Assume  $A, B, C$  is nonbipartite and normalized. Then  $\alpha_i = \alpha'_i = \alpha''_i$  for  $0 \leq i \leq d$ . Moreover  $\varphi_i = \varphi'_i = \varphi''_i$  for  $1 \leq i \leq d$ .*

Let  $A, B, C$  be a normalized LR triple on  $V$  that is not  $q$ -Weyl type. Assume  $d \geq 2$ , and let (13) be the parameter array of  $A, B, C$ . By the classification in [6, Sections 26, 28, 29],  $A, B, C$  is isomorphic to one of the following three types of LR triples:

**Definition 6.5** (See [6, Example 28.1].) The LR triple  $\text{NBG}_d(\mathbb{F}; q)$  is over  $\mathbb{F}$ , diameter  $d$ , nonbipartite, normalized, and satisfies

$$\begin{aligned} d &\geq 2; & 0 &\neq q \in \mathbb{F}; \\ q &\neq 1 \quad (1 \leq i \leq d); & q^{d+1} &\neq -1; \\ \varphi_i &= \frac{q(q^i - 1)(q^{i-d-1} - 1)}{(q - 1)^2} & (1 \leq i \leq d). \end{aligned}$$

**Definition 6.6** (See [6, Example 28.2].) The LR triple  $\text{NBG}_d(\mathbb{F}; 1)$  is over  $\mathbb{F}$ , diameter  $d$ , nonbipartite, normalized, and satisfies

$$\begin{aligned} d &\geq 2; & \text{Char}(\mathbb{F}) &\text{ is 0 or greater than } d; \\ \varphi_i &= i(i - d - 1) & (1 \leq i \leq d). \end{aligned}$$

**Definition 6.7** (See [6, Example 29.1].) The LR triple  $\text{NBNG}_d(\mathbb{F}; t)$  is over  $\mathbb{F}$ , diameter  $d$ , nonbipartite, normalized, and satisfies

$$\begin{aligned} d &\geq 4; & d &\text{ is even}; & 0 &\neq t \in \mathbb{F}; \\ t^i &\neq 1 \quad (1 \leq i \leq d/2); & t^{d+1} &\neq 1; \\ \varphi_i &= \begin{cases} t^{i/2} - 1 & \text{if } i \text{ is even,} \\ t^{(i-d-1)/2} - 1 & \text{if } i \text{ is odd} \end{cases} & (1 \leq i \leq d). \end{aligned}$$

## 7 Bounding the dimension of the tridiagonal space; nonbipartite case

Let  $A, B, C$  be an LR triple on  $V$  with parameter array (13), Toeplitz data (15), and idempotent data (1). For  $a, q \in \mathbb{F}$  and an integer  $n \geq 0$ , define

$$(a; q)_n = (1 - a)(1 - aq) \cdots (1 - aq^{n-1}).$$

We interpret  $(a; q)_0 = 1$ .

**Lemma 7.1** (See [6, Proposition 31.3]) *The following hold:*

(i) *For the LR triple  $\text{NBG}_d(\mathbb{F}; q)$ ,*

$$\alpha_i = \frac{(1-q)^i}{(q; q)_i} \quad (0 \leq i \leq d).$$

(ii) *For the LR triple  $\text{NBG}_d(\mathbb{F}; 1)$ ,*

$$\alpha_i = \frac{1}{i!} \quad (0 \leq i \leq d).$$

(iii) *For the LR triple  $\text{NBNG}_d(\mathbb{F}; t)$ ,*

$$\alpha_i = \begin{cases} \frac{1}{(t; t)_{i/2}} & \text{if } i \text{ is even,} \\ \frac{1}{(t; t)_{(i-1)/2}} & \text{if } i \text{ is odd} \end{cases} \quad (0 \leq i \leq d).$$

Assume  $A, B, C$  is one of  $\text{NBG}_d(\mathbb{F}; q)$ ,  $\text{NBG}_d(\mathbb{F}; 1)$ ,  $\text{NBNG}_d(\mathbb{F}; t)$ . Let  $\mathcal{X}$  be the tridiagonal space for  $A, B, C$ .

**Lemma 7.2** *The following hold:*

(i)  $\alpha_i \neq 0$  for  $0 \leq i \leq d$ .

(ii)  $\alpha_i^2 \varphi_i \neq \alpha_{i-1} \alpha_{i+1} \varphi_{i+1}$  for  $1 \leq i \leq d-1$ .

**Proof.** (i): Follows from Lemma 7.1.

(ii): We show

$$\frac{\alpha_i \varphi_i}{\alpha_{i-1}} - \frac{\alpha_{i+1} \varphi_{i+1}}{\alpha_i} \neq 0 \quad (1 \leq i \leq d-1). \quad (17)$$

First consider  $\text{NBG}_d(\mathbb{F}; q)$ . Using Definition 6.5 and Lemma 7.1(i), one checks

$$\frac{\alpha_i \varphi_i}{\alpha_{i-1}} - \frac{\alpha_{i+1} \varphi_{i+1}}{\alpha_i} = -q^{i-d} \quad (1 \leq i \leq d-1).$$

So (17) holds. Next consider  $\text{NBG}_d(\mathbb{F}; 1)$ . Using Definition 6.6 and Lemma 7.1(ii), one checks

$$\frac{\alpha_i \varphi_i}{\alpha_{i-1}} - \frac{\alpha_{i+1} \varphi_{i+1}}{\alpha_i} = -1 \quad (1 \leq i \leq d-1).$$

So (17) holds. Next consider  $\text{NBNG}_d(\mathbb{F}; t)$ . Using Definition 6.7 and Lemma 7.1(iii), one checks

$$\frac{\alpha_i \varphi_i}{\alpha_{i-1}} - \frac{\alpha_{i+1} \varphi_{i+1}}{\alpha_i} = \begin{cases} -t^{(i-d)/2} & \text{if } i \text{ is even,} \\ t^{(i-d-1)/2} & \text{if } i \text{ is odd} \end{cases} \quad (1 \leq i \leq d-1).$$

So (17) holds. The result follows.  $\square$

**Lemma 7.3** *Let  $X \in \mathcal{X}$  such that*

$$XE'_d = 0, \quad XAE'_d = 0, \quad E''_d X = 0.$$

*Then  $X = 0$ .*

**Proof.** Note that  $\alpha_i \neq 0$  for  $0 \leq i \leq d$  by Lemma 7.2(i). Fix an  $(A, B)$ -basis  $\{v_i\}_{i=0}^d$  for  $V$ . We identify each element of  $\text{End}(V)$  with the matrix that represent it with respect to  $\{v_i\}_{i=0}^d$ . By Propositions 4.7, 4.8, and Lemmas 3.5, 6.4,

$$\begin{aligned} (E'_d)_{i,0} &= \alpha_i \varphi_1 \cdots \varphi_i & (0 \leq i \leq d), \\ (E''_d)_{0,j} &= \alpha_j & (0 \leq j \leq d). \end{aligned}$$

Observe that  $X$  is tridiagonal since  $X \in \mathcal{X}$ . For  $0 \leq i \leq d$  let  $x_i$  be the  $(i, i)$ -entry of  $X$ . For  $1 \leq i \leq d$  let  $y_{i-1}$  be the  $(i-1, i)$ -entry of  $X$ , and let  $z_i$  be the  $(i, i-1)$ -entry of  $X$ :

$$X = \begin{pmatrix} x_0 & y_0 & & & \mathbf{0} \\ z_1 & x_1 & y_1 & & \\ & z_2 & \cdot & \cdot & \\ & & \cdot & \cdot & \\ \mathbf{0} & & & \cdot & \cdot & y_{d-1} \\ & & & z_d & x_d \end{pmatrix}.$$

For  $0 \leq i \leq d$ , compute the  $(i, 0)$ -entry of  $XE'_d$  to find

$$0 = \alpha_0 x_0 + \alpha_1 \varphi_1 y_0, \quad (18)$$

$$0 = \alpha_i x_i + \alpha_{i+1} \varphi_{i+1} y_i + \alpha_{i-1} \varphi_i^{-1} z_i \quad (1 \leq i \leq d-1), \quad (19)$$

$$0 = \alpha_d x_d + \alpha_{d-1} \varphi_d^{-1} z_d. \quad (20)$$

By Lemma 2.4, for  $0 \leq i, j \leq d$ ,  $A_{i,j} = 1$  if  $j = i+1$  and  $A_{i,j} = 0$  if  $j \neq i+1$ . Using this, for  $0 \leq i \leq d$ , compute the  $(i, 0)$ -entry of  $XAE'_d$  to find

$$0 = \alpha_1 x_0 + \alpha_2 \varphi_2 y_0, \quad (21)$$

$$0 = \alpha_{i+1} x_i + \alpha_{i+2} \varphi_{i+2} y_i + \alpha_i \varphi_{i+1}^{-1} z_i \quad (1 \leq i \leq d-2), \quad (22)$$

$$0 = \alpha_d x_{d-1} + \alpha_{d-1} \varphi_d^{-1} z_{d-1}, \quad (23)$$

$$0 = \alpha_d z_d. \quad (24)$$

For  $0 \leq i \leq d-2$ , compute the  $(0, i)$ -entry of  $E''_d X$  to find

$$0 = \alpha_0 x_0 + \alpha_1 z_1, \quad (25)$$

$$0 = \alpha_i x_i + \alpha_{i-1} y_{i-1} + \alpha_{i+1} z_{i+1} \quad (1 \leq i \leq d-2). \quad (26)$$

In (19) for  $i = 1$  and (25), eliminate  $z_1$  to get

$$\alpha_1 x_1 + \alpha_2 \varphi_2 y_1 - \alpha_0^2 \alpha_1^{-1} \varphi_1^{-1} x_0 = 0. \quad (27)$$



In (19) and (26) with  $i \rightarrow i - 1$ , eliminate  $z_i$  to get

$$\alpha_i x_i + \alpha_{i+1} \varphi_{i+1} y_i - \alpha_{i-1}^2 \alpha_i^{-1} \varphi_i^{-1} x_{i-1} - \alpha_{i-2} \alpha_{i-1} \alpha_i^{-1} \varphi_i^{-1} y_{i-2} = 0 \quad (2 \leq i \leq d-1). \quad (28)$$

In (22) for  $i = 1$  and (25), eliminate  $z_1$  to get

$$\alpha_2 x_1 + \alpha_3 \varphi_3 y_1 - \alpha_0 \varphi_2^{-1} x_0 = 0. \quad (29)$$

In (22) and (26) with  $i \rightarrow i - 1$ , eliminate  $z_i$  to get

$$\alpha_{i+1} x_i + \alpha_{i+2} \varphi_{i+2} y_i - \alpha_{i-1} \varphi_{i+1}^{-1} x_{i-1} - \alpha_{i-2} \varphi_{i+1}^{-1} y_{i-2} = 0 \quad (2 \leq i \leq d-2). \quad (30)$$

We show that  $x_i = 0$  and  $y_i = 0$  for  $0 \leq i \leq d-2$  using induction on  $i$ . By Lemma 7.2(ii)  $\alpha_1^2 \varphi_1 - \alpha_0 \alpha_2 \varphi_2 \neq 0$ . By this and (18), (21) we get  $x_0 = 0$  and  $y_0 = 0$ . By this and (27), (29),

$$\alpha_1 x_1 + \alpha_2 \varphi_2 y_1 = 0, \quad (31)$$

$$\alpha_2 x_1 + \alpha_3 \varphi_3 y_1 = 0. \quad (32)$$

By Lemma 7.2(ii)  $\alpha_2^2 \varphi_2 - \alpha_1 \alpha_3 \varphi_3 \neq 0$ . By this and (31), (32) we get  $x_1 = 0$  and  $y_1 = 0$ . Assume  $2 \leq i \leq d-2$  and  $x_j = 0$ ,  $y_j = 0$  for  $j < i$ . By (28) and (30),

$$\alpha_i x_i + \alpha_{i+1} \varphi_{i+1} y_i = 0, \quad (33)$$

$$\alpha_{i+1} x_i + \alpha_{i+2} \varphi_{i+2} y_i = 0. \quad (34)$$

By Lemma 7.2(ii)  $\alpha_{i+1}^2 \varphi_{i+1} - \alpha_i \alpha_{i+2} \varphi_{i+2} \neq 0$ . By this and (33), (34) we get  $x_i = 0$  and  $y_i = 0$ . We have shown that  $x_i = 0$  and  $y_i = 0$  for  $0 \leq i \leq d-2$ . By this and (25), (26) we get  $z_i = 0$  for  $1 \leq i \leq d-1$ . By (23) we get  $x_{d-1} = 0$ . By (19) for  $i = d-1$  we get  $y_{d-1} = 0$ . By (24) we get  $z_d = 0$ . By (20) we get  $x_d = 0$ . We have shown that  $x_i = 0$  ( $0 \leq i \leq d$ ),  $y_i = 0$  ( $0 \leq i \leq d-1$ ) and  $z_i = 0$  ( $1 \leq i \leq d$ ). Thus  $X = 0$ .  $\square$

**Lemma 7.4** *We have*

$$\dim \mathcal{X} E'_d \leq 2, \quad \dim E''_d \mathcal{X} \leq 2, \quad \dim \mathcal{X} A E'_d \leq 3.$$

**Proof.** Abbreviate  $\mathcal{A} = \text{End}(V)$ . Observe that  $\dim E'_i \mathcal{A} E'_j = 1$  for  $0 \leq i, j \leq d$ . By the definition of  $\mathcal{X}$ ,  $E'_r \mathcal{X} E'_d = 0$  for  $0 \leq r \leq d-2$ . Using this we argue

$$\mathcal{X} E'_d = I \mathcal{X} E'_d = \sum_{r=0}^d E'_r \mathcal{X} E'_d = E'_{d-1} \mathcal{X} E'_d + E'_d \mathcal{X} E'_d.$$

So  $\mathcal{X} E'_d \subseteq E'_{d-1} \mathcal{A} E'_d + E'_d \mathcal{A} E'_d$ , and therefore  $\dim \mathcal{X} E'_d \leq 2$ . Similarly  $\dim E''_d \mathcal{X} \leq 2$ . By Lemma 2.3,  $A E'_d = E'_{d-1} \mathcal{A} E'_d + E'_d \mathcal{A} E'_d$ , so

$$A E'_d \in E'_{d-1} \mathcal{A} E'_d + E'_d \mathcal{A} E'_d.$$

Therefore

$$\mathcal{X}AE'_d \subseteq \mathcal{X}E'_{d-1}\mathcal{A}E'_d + \mathcal{X}E'_d\mathcal{A}E'_d.$$

By this and the definition of  $\mathcal{X}$ ,

$$\mathcal{X}AE'_d \subseteq E'_{d-2}\mathcal{A}E'_d + E'_{d-1}\mathcal{A}E'_d + E'_d\mathcal{A}E'_d.$$

Therefore  $\dim \mathcal{X}AE'_d \leq 3$ .  $\square$

**Lemma 7.5** *The space  $\mathcal{X}$  has dimension at most 7.*

**Proof.** Define linear maps  $\pi_1 : \mathcal{X} \rightarrow \mathcal{X}E'_d$  that sends  $X \in \mathcal{X}$  to  $XE'_d$ ,  $\pi_2 : \mathcal{X} \rightarrow E''_d\mathcal{X}$  that sends  $X \in \mathcal{X}$  to  $E''_dX$ ,  $\pi_3 : \mathcal{X} \rightarrow \mathcal{X}AE'_d$  that sends  $X \in \mathcal{X}$  to  $XAE'_d$ . For  $i = 1, 2, 3$  let  $K_i$  be the kernel of  $\pi_i$ . By Lemma 7.4 the image of  $\pi_1$  (resp.  $\pi_2$ ) (resp.  $\pi_3$ ) has dimension at most 2 (resp. 2) (resp. 3). Therefore  $K_1$  (resp.  $K_2$ ) (resp.  $K_3$ ) has codimension at most 2 (resp. 2) (resp. 3). By these comments the space  $K_1 \cap K_2 \cap K_3$  has codimension at most 7. On the other hand, by Lemma 7.3  $K_1 \cap K_2 \cap K_3 = 0$ . Therefore  $\mathcal{X}$  has dimension at most 7.  $\square$

## 8 Proof of Theorem 1.1

Let  $A, B, C$  be an LR triple on  $V$  with parameter array (13), Toeplitz data (15), trace data (2), and idempotent data (1).

**Lemma 8.1** (See [6, Proposition 14.1].) *For  $0 \leq i \leq d$ ,*

$$a_{d-i} = \alpha'_0\beta'_1\varphi''_i + \alpha'_1\beta'_0\varphi''_{i+1}.$$

Assume  $A, B, C$  is nonbipartite. In view of Corollary 5.2, assume that  $A, B, C$  is normalized. By Lemma 6.4

$$\alpha_i = \alpha'_i = \alpha''_i \quad (0 \leq i \leq d), \quad (35)$$

$$\varphi_i = \varphi'_i = \varphi''_i \quad (1 \leq i \leq d). \quad (36)$$

**Lemma 8.2** *For  $0 \leq i \leq d$ ,*

$$a_i = \varphi_{d-i+1} - \varphi_{d-i}.$$

**Proof.** By Lemma 8.1 and (35), (36),

$$a_i = \alpha_0\beta_1\varphi_i + \alpha_1\beta_0\varphi_{i+1}.$$

By Lemma 3.5  $\alpha_0 = 1$  and  $\beta_0 = 1$ . By Definition 6.2 and Lemma 3.5,  $\alpha_1 = 1$  and  $\beta_1 = -1$ . Now the result follows from these comments.  $\square$

**Proof of Theorem 1.1.** In view of Corollary 5.2 we may assume that  $A, B, C$  is normalized. So  $A, B, C$  is one of  $\text{NBG}_d(\mathbb{F}; q)$ ,  $\text{NBG}_d(\mathbb{F}; 1)$ ,  $\text{NBNG}_d(\mathbb{F}; t)$ .

First assume  $d \geq 3$ . By Lemma 7.5, it suffices to show that the elements (8) are linearly independent. For scalars  $e, f_1, f_2, f_3, g_1, g_2, g_3 \in \mathbb{F}$ , define

$$Y = eI + f_1A + f_2B + f_3C + g_1ABC + g_2ACB + g_3CAB. \quad (37)$$

Assume  $Y = 0$ , and we show that  $e, f_1, f_2, f_3, g_1, g_2, g_3$  are all 0. Fix an  $(A, B)$ -basis  $\{v_i\}_{i=0}^d$  for  $V$ . We identify each element of  $\text{End}(V)$  with the matrix representing it with respect to  $\{v_i\}_{i=0}^d$ . By Lemma 2.4 and (36), each of  $A, B, C$  is tridiagonal with the following entries:

$$\begin{array}{ccc|ccc|ccc} A_{i,i-1} & A_{i,i} & A_{i-1,i} & B_{i,i-1} & B_{i,i} & B_{i-1,i} & C_{i,i-1} & C_{i,i} & C_{i-1,i} \\ \hline 0 & 0 & 1 & \varphi_i & 0 & 0 & \varphi_{d-i+1} & a_i & \varphi_{d-i+1}/\varphi_i \end{array}$$

Using these entries together with Lemma 8.2, we compute the  $(i, j)$ -entry of  $Y$  for  $(i, j) = (0, 0), (0, 1), (1, 0), (1, 1), (1, 2), (2, 1), (2, 3)$ ; this yields a system of linear equations with unknowns  $e, f_1, f_2, f_3, g_1, g_2, g_3$ . The coefficient matrix  $M$  is as follows:

$(i, j)$	$e$	$f_1$	$f_2$	$f_3$	$g_1$	$g_2$	$g_3$
$(0, 0)$	1	0	0	$-\varphi_d$	$-\varphi_1\varphi_d$	$\varphi_1(\varphi_d - \varphi_{d-1})$	$-\varphi_1\varphi_d$
$(0, 1)$	0	1	0	$\varphi_1^{-1}\varphi_d$	$\varphi_d$	$\varphi_{d-1}$	$\varphi_1^{-1}\varphi_2\varphi_d$
$(1, 0)$	0	0	$\varphi_1$	$\varphi_d$	$\varphi_2\varphi_d$	$\varphi_1\varphi_{d-1}$	$\varphi_1\varphi_d$
$(1, 1)$	1	0	0	$\varphi_d - \varphi_{d-1}$	$\varphi_2(\varphi_d - \varphi_{d-1})$	$\varphi_2(\varphi_{d-1} - \varphi_{d-2})$	$\varphi_2(\varphi_d - \varphi_{d-1})$
$(1, 2)$	0	1	0	$\varphi_2^{-1}\varphi_{d-1}$	$\varphi_{d-1}$	$\varphi_{d-2}$	$\varphi_2^{-1}\varphi_3\varphi_{d-1}$
$(2, 1)$	0	0	$\varphi_2$	$\varphi_{d-1}$	$\varphi_3\varphi_{d-1}$	$\varphi_2\varphi_{d-2}$	$\varphi_2\varphi_{d-1}$
$(2, 3)$	0	1	0	$\varphi_3^{-1}\varphi_{d-2}$	$\varphi_{d-2}$	$\varphi_{d-3}$	$\varphi_3^{-1}\varphi_4\varphi_{d-2}$

One routinely compute the determinant of  $M$  for each of the cases  $\text{NBG}_d(\mathbb{F}; q)$ ,  $\text{NBG}_d(\mathbb{F}; 1)$ ,  $\text{NBNG}_d(\mathbb{F}; t)$ :

case	determinant of $M$
$\text{NBG}_d(\mathbb{F}; q)$	$\frac{(q^2 - 1)^2(q^{d-1} - 1)(q^d - 1)(q^{d+1} + 1)^3}{q^{2d+3}(q - 1)^4}$
$\text{NBG}_d(\mathbb{F}; 1)$	$32d(d - 1)$
$\text{NBNG}_d(\mathbb{F}; t)$	$-t^{3(d+2)/2}(t - 1)^2(t^{d/2} - 1)(t^{d+1} - 1)^3$

In each case, the determinant of  $M$  is nonzero, so  $e, f_1, f_2, f_3, g_1, g_2, g_3$  are all 0. The result follows.

Next assume  $d = 2$ . We proceed in a similar way as above. By Lemma 5.5 it suffices to show that the elements (7) are linearly independent. Define  $Y$  as in (37) with  $g_3 = 0$ . We compute the  $(i, j)$ -entry of  $Y$  for  $(i, j) = (0, 0), (0, 1), (1, 0), (1, 1), (1, 2), (2, 1)$ ; this yields a system of linear equations with unknowns  $e, f_1, f_2, f_3, g_1, g_2$ . The coefficient matrix  $M'$  is obtained from  $M$  by removing the last row and the last column, where we interpret

$\varphi_{d-2} = 0$  and  $\varphi_3 = 0$ . One routinely compute the determinant of  $M'$  for each of the cases  $\text{NBG}_2(\mathbb{F}; q)$ ,  $\text{NBG}_2(\mathbb{F}; 1)$ :

case	determinant of $M'$
$\text{NBG}_2(\mathbb{F}; q)$	$\frac{(q^2 - 1)^3(q^3 + 1)^2}{q^5(q - 1)^3}$
$\text{NBG}_2(\mathbb{F}; 1)$	32

In each case, the determinant of  $M'$  is nonzero, so  $e, f_1, f_2, f_3, g_1, g_2$  are all 0. The result follows.  $\square$

## 9 Bipartite LR triples

In this section, we recall the classification of bipartite LR triples. Let  $A, B, C$  be a bipartite LR triple on  $V$  with parameter array (13), Toeplitz data (15), and idempotent data (1). To avoid triviality, we assume  $d \geq 2$ .

**Lemma 9.1** (See [6, Lemma 16.6].) *The diameter  $d$  is even. Moreover for  $0 \leq i \leq d$ , each of  $\alpha_i, \alpha'_i, \alpha''_i, \beta_i, \beta'_i, \beta''_i$  is zero if  $i$  is odd and nonzero if  $i$  is even.*

By Lemma 9.1  $d$  is even; set  $m = d/2$ .

**Lemma 9.2** (See [6, Lemma 16.12].)

(i) *The following spaces are equal:*

$$\sum_{j=0}^m E_{2j}V, \quad \sum_{j=0}^m E'_{2j}V, \quad \sum_{j=0}^m E''_{2j}V. \quad (38)$$

(ii) *The following spaces are equal:*

$$\sum_{j=0}^{m-1} E_{2j+1}V, \quad \sum_{j=0}^{m-1} E'_{2j+1}V, \quad \sum_{j=0}^{m-1} E''_{2j+1}V. \quad (39)$$

Let  $V_{\text{out}}$  and  $V_{\text{in}}$  denote the common spaces of (38) and (39), respectively. Then

$$V = V_{\text{out}} + V_{\text{in}} \quad (\text{direct sum}).$$

**Lemma 9.3** (See [6, Lemma 16.15].) *We have*

$$\begin{aligned} AV_{\text{out}} &= V_{\text{in}}, & BV_{\text{out}} &= V_{\text{in}}, & CV_{\text{out}} &= V_{\text{in}}, \\ AV_{\text{in}} &\subseteq V_{\text{out}}, & BV_{\text{in}} &\subseteq V_{\text{out}}, & CV_{\text{in}} &\subseteq V_{\text{out}}. \end{aligned}$$

**Definition 9.4** (See [6, Definition 16.29].) Define  $A_{\text{out}}, A_{\text{in}}, B_{\text{out}}, B_{\text{in}}, C_{\text{out}}, C_{\text{in}}$  in  $\text{End}(V)$  as follows. The map  $A_{\text{out}}$  acts on  $V_{\text{out}}$  as  $A$ , and on  $V_{\text{in}}$  as zero. The map  $A_{\text{in}}$  acts on  $V_{\text{in}}$  as  $A$ , and on  $V_{\text{out}}$  as zero. The other maps are similarly defined.

Observe

$$A = A_{\text{out}} + A_{\text{in}}, \quad B = B_{\text{out}} + B_{\text{in}}, \quad C = C_{\text{out}} + C_{\text{in}}.$$

**Lemma 9.5** (See [6, Lemma 16.31, 16.33].) *For nonzero scalars  $\alpha_{\text{out}}, \alpha_{\text{in}}, \beta_{\text{out}}, \beta_{\text{in}}, \gamma_{\text{out}}, \gamma_{\text{in}}$  in  $\mathbb{F}$ , the sequence*

$$\alpha_{\text{out}}A_{\text{out}} + \alpha_{\text{in}}A_{\text{in}}, \quad \beta_{\text{out}}B_{\text{out}} + \beta_{\text{in}}B_{\text{in}}, \quad \gamma_{\text{out}}C_{\text{out}} + \gamma_{\text{in}}C_{\text{in}} \quad (40)$$

*is a bipartite LR triple on  $V$ . Moreover this LR triple has the same idempotent data as  $A, B, C$ .*

**Definition 9.6** (See [6, Definition 16.36].) Two bipartite LR triples  $A, B, C$  and  $A', B', C'$  are called *biassociate* whenever there exist nonzero scalars  $\alpha, \beta, \gamma$  such that

$$A' = \alpha A_{\text{out}} + A_{\text{in}}, \quad B' = \beta B_{\text{out}} + B_{\text{in}}, \quad C' = \gamma C_{\text{out}} + C_{\text{in}}.$$

**Lemma 9.7** *Biassociate bipartite LR triples have the same tridiagonal space.*

**Proof.** By Lemma 9.5 biassociate LR triples have the same idempotent data. The result follows.  $\square$

In the direct sum  $V = V_{\text{out}} + V_{\text{in}}$ , let  $J \in \text{End}(V)$  denote the projection onto  $V_{\text{out}}$ . Then  $I - J$  is the projection onto  $V_{\text{in}}$ . Observe  $J^2 = J$  and  $J(I - J) = 0 = (I - J)J$ . Also  $V_{\text{out}} = JV$  and  $V_{\text{in}} = (I - J)V$ .

**Lemma 9.8** (See [6, Lemma 16.30].) *We have*

$$\begin{aligned} A_{\text{out}} &= AJ = (I - J)A, & A_{\text{in}} &= JA = A(I - J), \\ B_{\text{out}} &= BJ = (I - J)B, & B_{\text{in}} &= JB = B(I - J), \\ C_{\text{out}} &= CJ = (I - J)C, & C_{\text{in}} &= JC = C(I - J). \end{aligned}$$

**Lemma 9.9** *We have*

$$\begin{aligned} ABCJ &= A_{\text{out}}B_{\text{in}}C_{\text{out}}, & ACBJ &= A_{\text{out}}C_{\text{in}}B_{\text{out}}, \\ BCAJ &= B_{\text{out}}C_{\text{in}}A_{\text{out}}, & BACJ &= B_{\text{out}}A_{\text{in}}C_{\text{out}}, \\ CABJ &= C_{\text{out}}A_{\text{in}}B_{\text{out}}, & BACJ &= C_{\text{out}}B_{\text{in}}A_{\text{out}}. \end{aligned}$$

**Proof.** Use  $J^2 = J$ ,  $(I - J)^2 = I - J$  and Lemma 9.8.  $\square$

**Lemma 9.10** For nonzero scalars  $\alpha_{\text{out}}, \alpha_{\text{in}}, \beta_{\text{out}}, \beta_{\text{in}}, \gamma_{\text{out}}, \gamma_{\text{in}}$  in  $\mathbb{F}$ , define

$$A' = \alpha_{\text{out}}A_{\text{out}} + \alpha_{\text{in}}A_{\text{in}}, \quad B' = \beta_{\text{out}}B_{\text{out}} + \beta_{\text{in}}B_{\text{in}}, \quad C' = \gamma_{\text{out}}C_{\text{out}} + \gamma_{\text{in}}C_{\text{in}}.$$

Then  $A'B'C'J$  is a nonzero scalar multiple of  $ABCJ$ , and  $A'C'B'J$  is a nonzero scalar multiple of  $ACBJ$ .

**Proof.** By Lemma 9.9  $A'B'C'J = A'_{\text{out}}B'_{\text{in}}C'_{\text{out}}$ . Observe  $A_{\text{out}}J = A_{\text{out}}$  and  $A_{\text{in}}J = 0$ . Using this we argue

$$A'_{\text{out}} = A'J = \alpha_{\text{out}}A_{\text{out}}J + \alpha_{\text{in}}A_{\text{in}}J = \alpha_{\text{out}}A_{\text{out}}.$$

So  $A'_{\text{out}} = \alpha_{\text{out}}A_{\text{out}}$ . Similarly  $B'_{\text{in}} = \beta_{\text{in}}B_{\text{in}}$ ,  $C'_{\text{out}} = \gamma_{\text{out}}C_{\text{out}}$ . By these comments

$$A'B'C'J = \alpha_{\text{out}}\beta_{\text{in}}\gamma_{\text{out}}A_{\text{out}}B_{\text{in}}C_{\text{out}}.$$

By this and Lemma 9.9  $A'B'C'J = \alpha_{\text{out}}\beta_{\text{in}}\gamma_{\text{out}}ABCJ$ . So  $A'B'C'J$  is a nonzero scalar multiple of  $ABCJ$ . The proof is similar for  $A'C'B'J$ .  $\square$

We recall the normalization of a bipartite LR triple.

**Definition 9.11** See [6, Definition 18.11].) The LR triple  $A, B, C$  is said to be *normalized* whenever  $\alpha_2 = 1$ ,  $\alpha'_2 = 1$ ,  $\alpha''_2 = 1$ .

**Lemma 9.12** See [6, Corollary 18.15].) *There exists a unique sequence of nonzero scalars  $\alpha, \beta, \gamma$  in  $\mathbb{F}$  such that the LR triple  $\alpha A_{\text{out}} + A_{\text{in}}, \beta B_{\text{out}} + B_{\text{in}}, \gamma C_{\text{out}} + C_{\text{in}}$  is normalized.*

**Lemma 9.13** (See [6, Lemma 18.12].) *Assume  $A, B, C$  is bipartite and normalized. Then  $\alpha_i = \alpha'_i = \alpha''_i$  and  $\beta_i = \beta'_i = \beta''_i$  for  $0 \leq i \leq d$ .*

Assume  $A, B, C$  is normalized. By the classification in [6, Section 39],  $A, B, C$  is isomorphic to one of the following LR triples.

**Definition 9.14** (See [6, Example 30.1].) The LR triple  $B_d(\mathbb{F}; t, \rho_0, \rho'_0, \rho''_0)$  is over  $\mathbb{F}$ , diameter  $d$ , bipartite, normalized, and satisfies

$$\begin{aligned} d &\geq 4; & d &\text{ is even; } & 0 &\neq t \in \mathbb{F}; & t^i &\neq 1 \quad (1 \leq i \leq d/2); \\ \rho_0, \rho'_0, \rho''_0 &\in \mathbb{F}; & \rho_0 \rho'_0 \rho''_0 &= -t^{1-d/2}; \\ \varphi_i &= \begin{cases} \rho_0 \frac{1-t^{i/2}}{1-t} & \text{if } i \text{ is even,} \\ \frac{t}{\rho_0} \frac{1-t^{(i-d-1)/2}}{1-t} & \text{if } i \text{ is odd} \end{cases} & (1 \leq i \leq d); \\ \varphi'_i &= \begin{cases} \rho'_0 \frac{1-t^{i/2}}{1-t} & \text{if } i \text{ is even,} \\ \frac{t}{\rho'_0} \frac{1-t^{(i-d-1)/2}}{1-t} & \text{if } i \text{ is odd} \end{cases} & (1 \leq i \leq d); \\ \varphi''_i &= \begin{cases} \rho''_0 \frac{1-t^{i/2}}{1-t} & \text{if } i \text{ is even,} \\ \frac{t}{\rho''_0} \frac{1-t^{(i-d-1)/2}}{1-t} & \text{if } i \text{ is odd} \end{cases} & (1 \leq i \leq d). \end{aligned}$$

**Definition 9.15** (See [6, Example 30.2].) The LR triple  $B_d(\mathbb{F}; 1, \rho_0, \rho'_0, \rho''_0)$  is over  $\mathbb{F}$ , diameter  $d$ , bipartite, normalized, and satisfies

$$\begin{aligned} d &\geq 4; & d \text{ is even}; & \text{Char}(\mathbb{F}) \text{ is 0 or greater than } d/2; \\ \rho_0, \rho'_0, \rho''_0 &\in \mathbb{F}; & \rho_0 \rho'_0 \rho''_0 &= -1; \\ \varphi_i &= \begin{cases} \frac{i\rho_0}{2} & \text{if } i \text{ is even,} \\ \frac{i-d-1}{2\rho_0} & \text{if } i \text{ is odd} \end{cases} & (1 \leq i \leq d); \\ \varphi'_i &= \begin{cases} \frac{i\rho'_0}{2} & \text{if } i \text{ is even,} \\ \frac{i-d-1}{2\rho'_0} & \text{if } i \text{ is odd} \end{cases} & (1 \leq i \leq d); \\ \varphi''_i &= \begin{cases} \frac{i\rho''_0}{2} & \text{if } i \text{ is even,} \\ \frac{i-d-1}{2\rho''_0} & \text{if } i \text{ is odd} \end{cases} & (1 \leq i \leq d). \end{aligned}$$

**Definition 9.16** (See [6, Example 30.2].) The LR triple  $B_2(\mathbb{F}; \rho_0, \rho'_0, \rho''_0)$  is over  $\mathbb{F}$ , diameter 2, bipartite, normalized, and satisfies

$$\begin{aligned} \rho_0, \rho'_0, \rho''_0 &\in \mathbb{F}; & \rho_0 \rho'_0 \rho''_0 &= -1; \\ \varphi_1 &= -1/\rho_0, & \varphi'_1 &= -1/\rho'_0, & \varphi''_1 &= -1/\rho''_0, \\ \varphi_2 &= \rho_0, & \varphi'_2 &= \rho'_0, & \varphi''_2 &= \rho''_0. \end{aligned}$$

## 10 Bounding the dimension of the tridiagonal space; bipartite case

Let  $A, B, C$  be a bipartite LR triple on  $V$  with parameter array (13), Toeplitz data (15), and idempotent data (1). Let  $\mathfrak{X}$  be the tridiagonal space for  $A, B, C$ .

**Lemma 10.1** (See [6, Proposition 31.3]) *The following hold:*

(i) *For the LR triple  $B_d(\mathbb{F}; t, \rho_0, \rho'_0, \rho''_0)$ ,*

$$\alpha_{2i} = \frac{(1-t)^i}{(t; t)_i}, \quad \beta_{2i} = \frac{(-1)^i t^{i(i-1)/2} (1-t)^i}{(t; t)_i} \quad (0 \leq i \leq d/2).$$

(ii) *For the LR triple  $B_d(\mathbb{F}; 1, \rho_0, \rho'_0, \rho''_0)$ ,*

$$\alpha_{2i} = \frac{1}{i!}, \quad \beta_{2i} = \frac{(-1)^i}{i!} \quad (0 \leq i \leq d/2).$$

Assume  $A, B, C$  is normalized. We may assume that  $A, B, C$  is one of  $B_d(\mathbb{F}; t, \rho_0, \rho'_0, \rho''_0)$ ,  $B_d(\mathbb{F}; 1, \rho_0, \rho'_0, \rho''_0)$ ,  $B_2(\mathbb{F}; \rho_0, \rho'_0, \rho''_0)$ .

**Lemma 10.2** *We have*

$$\alpha_{2i}^2 \neq \alpha_{2i-2} \alpha_{2i+2}, \quad \beta_{2i}^2 \neq \beta_{2i-2} \beta_{2i+2} \quad (1 \leq i \leq d/2 - 1).$$

**Proof.** Use Lemma 10.1. □

**Lemma 10.3** Assume  $d \geq 4$ . Then the vector space  $\mathcal{X}$  has dimension at most 8.

**Proof.** Fix an  $(A, B)$ -basis  $\{v_i\}_{i=0}^d$  for  $V$ . We identify each element of  $\text{End}(V)$  with the matrix that represents it with respect to  $\{v_i\}_{i=0}^d$ . Pick an element  $X \in \mathcal{X}$ . Then  $X$  is tridiagonal; set

$$X = \begin{pmatrix} x_0 & y_0 & & & & \mathbf{0} \\ z_1 & x_1 & y_1 & & & \\ & z_2 & \cdot & \cdot & & \\ & & \cdot & \cdot & \cdot & \\ & & & \cdot & \cdot & y_{d-1} \\ \mathbf{0} & & & & z_d & x_d \end{pmatrix}.$$

For notational convenience, define  $y_{-1} = 0$ ,  $y_d = 0$ ,  $z_0 = 0$ ,  $z_{d+1} = 0$ . By Proposition 4.8 and Lemma 9.13,

$$(E_r'')_{i,j} = \begin{cases} \alpha_{r-d+j}\beta_{d-r-i} & \text{if } i \leq d-r \leq j, \\ 0 & \text{otherwise} \end{cases} \quad (0 \leq i, j \leq d).$$

For  $0 \leq r \leq d-2$ , compute the  $(0, d)$ -entry of  $E_{d-r}'' X E_{d-r-2}''$  if  $r$  is even, and compute the  $(1, d-1)$ -entry of  $E_{d-r}'' X E_{d-r-2}''$  if  $r$  is odd. This yields

$$\alpha_0 \beta_2 x_r + \alpha_2 \beta_0 x_{r+2} = 0 \quad (0 \leq r \leq d-2). \quad (41)$$

For  $0 \leq r \leq d-3$  we compute the  $(0, d-1)$ -entry of  $E_{d-r}'' X E_{d-r-3}''$  if  $r$  is even, and  $(1, d)$ -entry of  $E_{d-r}'' X E_{d-r-3}''$  if  $r$  is odd. This yields

$$\alpha_0 \beta_2 y_r + \alpha_2 \beta_0 y_{r+2} + \alpha_0 \beta_4 z_r + \alpha_2 \beta_2 z_{r+2} + \alpha_4 \beta_0 z_{r+4} = 0 \quad (0 \leq r \leq d-3). \quad (42)$$

When  $d \geq 6$ , for  $0 \leq r \leq d-5$  we compute the  $(0, d-1)$ -entry of  $E_{d-r}'' X E_{d-r-5}''$  if  $r$  is even, and  $(1, d)$ -entry of  $E_{d-r}'' X E_{d-r-5}''$  if  $r$  is odd. This yields

$$\begin{aligned} & \alpha_0 \beta_4 y_r + \alpha_2 \beta_2 y_{r+2} + \alpha_4 \beta_0 y_{r+4} \\ & + \alpha_0 \beta_6 z_r + \alpha_2 \beta_4 z_{r+2} + \alpha_4 \beta_2 z_{r+4} + \alpha_6 \beta_0 z_{r+6} = 0 \quad (0 \leq r \leq d-5). \end{aligned} \quad (43)$$

We show that each entry of  $X$  are uniquely determined by  $x_0, x_1, y_0, y_{d-1}, z_2, z_4, z_{d-3}, z_{d-1}$ . To this aim, we assume

$$x_0 = 0, \quad x_1 = 0, \quad y_0 = 0, \quad y_{d-1} = 0, \quad z_2 = 0, \quad z_4 = 0, \quad z_{d-3} = 0, \quad z_{d-1} = 0, \quad (44)$$

and we show  $X = 0$ . By (41) and  $x_0 = 0, x_1 = 0$  we find  $x_r = 0$  for  $0 \leq r \leq d$ . We show that

$$y_{d-r} = 0, \quad z_{d-r} = 0 \quad r = 1, 3, 5, \dots, d-1. \quad (45)$$

By (44)  $y_{d-1} = 0$  and  $z_{d-1} = 0$ , so (45) holds for  $r = 1$ . By (42) for  $r = d-3$ ,

$$\alpha_0 \beta_2 y_{d-3} + \alpha_2 \beta_0 y_{d-1} + \alpha_0 \beta_4 z_{d-3} + \alpha_2 \beta_2 z_{d-1} = 0.$$



By (44)  $z_{d-3} = 0$ ,  $z_{d-1} = 0$ ,  $y_{d-1} = 0$ . By these comments  $y_{d-3} = 0$ . So (45) holds for  $r = 3$ . Now assume  $r \geq 5$ ,  $r$  is odd. By induction and (42),

$$\alpha_0\beta_2y_r + \alpha_0\beta_4z_r = 0. \quad (46)$$

By induction and (43),

$$\alpha_0\beta_4y_r + \alpha_0\beta_6z_r = 0. \quad (47)$$

By Lemma 10.2  $\beta_2\beta_6 - \beta_4^2 \neq 0$ , so we can solve the system of linear equations (46), (47); This yields  $y_r = 0$  and  $z_r = 0$ . We have shown (45). Next we show

$$y_r = 0, \quad z_{r+2} = 0 \quad r = 0, 2, 4, \dots, d-2. \quad (48)$$

By (44)  $y_0 = 0$  and  $z_2 = 0$ . So (48) holds for  $r = 0$ . By (42) for  $r = 0$ ,

$$\alpha_0\beta_2y_0 + \alpha_2\beta_0y_2 + \alpha_2\beta_2z_2 + \alpha_4\beta_0z_4 = 0.$$

By this and (44)  $y_2 = 0$ . So (48) holds for  $r = 2$ . Now assume  $r \geq 4$ ,  $r$  is even. By (42)

$$\alpha_0\beta_2y_r + \alpha_2\beta_0y_{r+2} + \alpha_0\beta_4z_r + \alpha_2\beta_2z_{r+2} + \alpha_4\beta_0z_{r+4} = 0.$$

By induction,  $y_r = 0$ ,  $z_r = 0$ ,  $z_{r+2} = 0$ . By these comments

$$\alpha_2\beta_0y_{r+2} + \alpha_4\beta_0z_{r+4} = 0. \quad (49)$$

By (43) with  $r \rightarrow r-2$ ,

$$\alpha_0\beta_4y_{r-2} + \alpha_2\beta_2y_r + \alpha_4\beta_0y_{r+2} + \alpha_0\beta_6z_{r-2} + \alpha_2\beta_4z_r + \alpha_4\beta_2z_{r+2} + \alpha_6\beta_0z_{r+4} = 0.$$

By induction  $y_{r-2} = 0$ ,  $y_r = 0$ ,  $z_{r-2} = 0$ ,  $z_r = 0$ ,  $z_{r+2} = 0$ . By these comments

$$\alpha_4\beta_0y_{r+2} + \alpha_6\beta_0z_{r+4} = 0. \quad (50)$$

By Lemma 10.2  $\alpha_2\alpha_6 - \alpha_4^2 \neq 0$ , so we can solve the linear equations (49), (50). This yields  $y_{r+2} = 0$  and  $z_{r+4} = 0$ . We have shown (48). By (44), (45), (48) we get  $y_r = 0$  for  $0 \leq r \leq d-1$  and  $z_r = 0$  for  $1 \leq r \leq d$ . So  $X = 0$ . Thus  $\mathcal{X}$  has dimension at most 8.  $\square$

## 11 Proof of Theorem 1.2

Let  $A, B, C$  be an LR triple on  $V$  with parameter array (13) and Toeplitz data (15).

**Lemma 11.1** (See [6, Proposition 14.6].) *For  $2 \leq i \leq d-1$ ,*

$$\begin{aligned} \frac{\varphi'_i}{\varphi''_{d-i+1}} &= \alpha'_0\beta'_2\varphi'_{i-1} + \alpha'_1\beta'_1\varphi'_i + \alpha'_2\beta'_0\varphi'_{i+1}, \\ \frac{\varphi''_i}{\varphi'_{d-i+1}} &= \alpha''_0\beta''_2\varphi'_{i-1} + \alpha''_1\beta''_1\varphi'_i + \alpha''_2\beta''_0\varphi'_{i+1}, \\ \frac{\varphi_i}{\varphi'_{d-i+1}} &= \alpha_0\beta_2\varphi''_{i-1} + \alpha_1\beta_1\varphi''_i + \alpha_2\beta_0\varphi''_{i+1}. \end{aligned}$$

**Lemma 11.2** ([6, Lemma 16.9].) *Assume  $A, B, C$  is bipartite and  $d \geq 2$ . Then  $\beta_2 = -\alpha_2$ ,  $\beta'_2 = -\alpha'_2$ ,  $\beta''_2 = -\alpha''_2$ .*

**Lemma 11.3** *Assume  $A, B, C$  is bipartite. Then*

$$\varphi_{i-1} \neq \varphi_{i+1}, \quad \varphi'_{i-1} \neq \varphi'_{i+1}, \quad \varphi''_{i-1} \neq \varphi''_{i+1} \quad (2 \leq i \leq d-1).$$

**Proof.** We first show  $\varphi_{i-1} \neq \varphi_{i+1}$ . By Lemma 3.5  $\alpha'_0 = 1$  and  $\beta'_0 = 1$ . By Lemma 9.1  $\alpha'_1 = 0$  and  $\beta'_1 = 0$ . By Lemma 11.2  $\beta'_2 = -\alpha'_2$ . By these comments and Lemma 11.1  $\beta'_2(\varphi_{i-1} - \varphi_{i+1}) = \varphi'_i/\varphi'_{d-i+1}$ . By Lemma 9.1  $\beta'_2 \neq 0$ . By these comments  $\varphi_{i-1} \neq \varphi_{i+1}$ . In a similar way, we can show that  $\varphi'_{i-1} \neq \varphi'_{i+1}$  and  $\varphi''_{i-1} \neq \varphi''_{i+1}$ .  $\square$

Let  $\mathcal{X}$  be the tridiagonal space for  $A, B, C$ .

**Lemma 11.4** *Assume  $A, B, C$  is bipartite. Then for  $X \in \mathcal{X}$ , the elements  $XJ$  and  $X(I-J)$  are contained in  $\mathcal{X}$ .*

**Proof.** Pick any  $X \in \mathcal{X}$ . Pick any integers  $r, s$  such that  $0 \leq r, s \leq d$  and  $|r-s| > 1$ . We show that  $E_r X J E_s = 0$ . By (3)  $J = \sum_{i=0}^{d/2} E_{2i}$ . So  $J E_s = 0$  if  $s$  is odd, and  $J E_s = E_s$  if  $s$  is even. We have  $E_r X E_s = 0$  since  $X \in \mathcal{X}$ . By these comments  $E_r X J E_s = 0$ . In a similar way,  $E'_r X J E'_s = 0$  and  $E''_r X J E''_s = 0$ . Thus  $XJ$  is contained in  $\mathcal{X}$ . Similarly,  $X(I-J)$  is contained in  $\mathcal{X}$ .  $\square$

**Proof of Theorem 1.2.** In view of Lemmas 9.7 and 9.10, we may assume that  $A, B, C$  is normalized.

Note by Lemma 11.4 that  $\mathcal{X}J \subseteq \mathcal{X}$  and  $\mathcal{X}(I-J) \subseteq \mathcal{X}$ . Observe  $X = XJ + X(I-J)$  for  $X \in \mathcal{X}$ . Therefore  $\mathcal{X} = \mathcal{X}J + \mathcal{X}(I-J)$ . Using  $J^2 = J$  and  $(I-J)J = 0$ , one checks that spaces  $\mathcal{X}J$  and  $\mathcal{X}(I-J)$  has zero intersection. So  $\mathcal{X} = \mathcal{X}J + \mathcal{X}(I-J)$  is a direct sum.

Fix an  $(A, B)$ -basis  $\{v_i\}_{i=0}^d$  for  $V$ . We identify each element of  $\text{End}(V)$  with the matrix representing it with respect to  $\{v_i\}_{i=0}^d$ . By Lemma 2.4 and since  $A, B, C$  is bipartite, each of  $A, B, C$  is tridiagonal with the following entries:

$$\begin{array}{ccc|ccc|ccc} A_{i,i-1} & A_{i,i} & A_{i-1,i} & B_{i,i-1} & B_{i,i} & B_{i-1,i} & C_{i,i-1} & C_{i,i} & C_{i-1,i} \\ \hline 0 & 0 & 1 & \varphi_i & 0 & 0 & \varphi''_{d-i+1} & 0 & \varphi'_{d-i+1}/\varphi_i \end{array}$$

The matrix  $J$  is the diagonal matrix whose  $(i, i)$ -entry is 1 if  $i$  is even, and 0 if  $i$  is odd.

(i): Assume  $d = 2$ . We have

$$J = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 & 0 \\ \varphi_1 & 0 & 0 \\ 0 & \varphi_2 & 0 \end{pmatrix}.$$

Now one routinely checks that  $J, AJ, BJ$  are linearly independent. Similarly  $I-J, A(I-J), B(I-J)$  are linearly independent. By Lemma 5.5  $\dim \mathcal{X} \leq 6$ . The result follows.

(ii): Assume  $d \geq 4$ . We first show that  $J, AJ, BJ, ACBJ$  are linearly independent. For scalars  $f_0, f_1, f_2, f_3$  in  $\mathbb{F}$ , set

$$Y = f_0J + f_1AJ + f_2BJ + f_3ACBJ.$$

We assume  $Y = 0$ , and we show  $f_0, f_1, f_2, f_3$  are all 0. Compute the  $(0,0)$ -entry of  $Y$  to find  $f_0 = 0$ . Compute the  $(i,j)$ -entry of  $Y$  for  $(i,j) = (1,0), (1,2), (3,2)$  to get

$$\begin{aligned} 0 &= f_2 + \varphi''_{d-1}f_3, \\ 0 &= f_1 + \varphi'_{d-2}f_3, \\ 0 &= f_2 + \varphi''_{d-3}f_3. \end{aligned}$$

Viewing the above equations as a system of linear equations with unknowns  $f_1, f_2, f_3$ , let  $M$  be the coefficient matrix. Then

$$M = \begin{pmatrix} 0 & 1 & \varphi''_{d-1} \\ 1 & 0 & \varphi'_{d-2} \\ 0 & 1 & \varphi''_{d-3} \end{pmatrix}.$$

The determinant of  $M$  is

$$\det M = \varphi''_{d-1} - \varphi''_{d-3}.$$

This is nonzero by Lemma 11.3, so  $f_1, f_2, f_3$  are all 0. We have shown that  $J, AJ, BJ, ACBJ$  are linearly independent.

Next we show that  $I - J, A(I - J), B(I - J), ABC(I - J)$  are linearly independent. For scalars  $f'_0, f'_1, f'_2, f'_3$  in  $\mathbb{F}$ , set

$$Y' = f'_0(I - J) + f'_1A(I - J) + f'_2B(I - J) + f'_3ABC(I - J).$$

We assume  $Y' = 0$ , and we show  $f'_0, f'_1, f'_2, f'_3$  are all 0. Compute the  $(1,1)$ -entry of  $Y'$  to find  $f'_0 = 0$ . Compute the  $(i,j)$ -entry of  $Y'$  for  $(i,j) = (0,1), (2,1), (2,3)$  to get

$$\begin{aligned} 0 &= f'_1 + \varphi'_df'_3, \\ 0 &= \varphi_2f'_2 + \varphi_3\varphi''_{d-1}f'_3, \\ 0 &= f'_1 + \varphi'_{d-2}f'_3. \end{aligned}$$

Viewing the above equations as a system of linear equations with unknowns  $f'_1, f'_2, f'_3$ , let  $M'$  be the coefficient matrix. Then

$$M' = \begin{pmatrix} 1 & 0 & \varphi'_d \\ 0 & \varphi_2 & \varphi_3\varphi''_{d-1} \\ 1 & 0 & \varphi'_{d-2} \end{pmatrix}.$$

The determinant of  $M'$  is

$$\det M' = \varphi_2(\varphi'_{d-2} - \varphi'_d).$$

This is nonzero by Lemma 11.3, so  $f'_1, f'_2, f'_3$  are all 0. We have shown that  $I - J, A(I - J), B(I - J), ABC(I - J)$  are linearly independent. By Lemma 10.3  $\dim \mathfrak{X} \leq 8$ . The result follows.  $\square$

## 12 Acknowledgement

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## 13 Appendix 1

Assume  $d \geq 3$ . Let  $A, B, C$  be a nonbipartite normalized LR triple that is not  $q$ -Weyl type. By Theorem 1.1 the elements

$$I, \quad A, \quad B, \quad C, \quad ABC, \quad ACB, \quad CAB$$

form a basis for the tridiagonal space  $\mathfrak{X}$ . We represent the elements  $BCA, BAC, CBA$  as a linear combination of the basis vectors. Below we give the coefficients of the linear combination.

For  $\text{NBG}_d(\mathbb{F}; q)$ :

	$I$	$A$	$B$	$C$	$ABC$	$ACB$	$CAB$
$BCA$	$\frac{(q^d-1)(q^{d+2}-1)}{q^d(q-1)^2}$	$-\frac{q^2-1}{q-1}$	$-\frac{q^2-1}{q-1}$	$\frac{(q^2-1)^2}{q(q-1)^2}$	$\frac{q^2-1}{q(q-1)}$	$-(2q+1)$	$\frac{q^2-1}{q(q-1)}$
$BAC$	$\frac{(q^d-1)(q^{d+2}-1)}{q^{d+1}(q-1)^2}$	$0$	$-\frac{q^2-1}{q(q-1)}$	$\frac{q^2-1}{q^2(q-1)}$	$\frac{q^2-1}{q^2(q-1)}$	$-\frac{q^2-1}{q(q-1)}$	$\frac{1}{q^2}$
$CBA$	$\frac{(q^d-1)(q^{d+2}-1)}{q^{d+1}(q-1)^2}$	$-\frac{q^2-1}{q(q-1)}$	$0$	$\frac{q^2-1}{q^2(q-1)}$	$\frac{1}{q^2}$	$-\frac{q^2-1}{q(q-1)}$	$\frac{q^2-1}{q^2(q-1)}$

For  $\text{NBG}_d(\mathbb{F}; 1)$ :

	$I$	$A$	$B$	$C$	$ABC$	$ACB$	$CAB$
$BCA$	$d(d+2)$	$-2$	$-2$	$4$	$2$	$-3$	$2$
$BAC$	$d(d+2)$	$0$	$-2$	$2$	$2$	$-2$	$1$
$CBA$	$d(d+2)$	$-2$	$0$	$2$	$1$	$-2$	$2$

For  $\text{NBNG}_d(\mathbb{F}; t)$ :

	$I$	$A$	$B$	$C$	$ABC$	$ACB$	$CAB$
$BCA$	$\frac{(t^{d/2}-1)(t^{(d+2)/2}-1)}{t^{d/2}}$	$t-1$	$t-1$	$0$	$0$	$t$	$0$
$BAC$	$-\frac{(t^{d/2}-1)(t^{(d+2)/2}-1)}{t^{(d+2)/2}}$	$0$	$-\frac{t-1}{t}$	$-\frac{t-1}{t}$	$0$	$0$	$\frac{1}{t}$
$CBA$	$-\frac{(t^{d/2}-1)(t^{(d+2)/2}-1)}{t^{(d+2)/2}}$	$-\frac{t-1}{t}$	$0$	$-\frac{t-1}{t}$	$\frac{1}{t}$	$0$	$0$

## 14 Appendix 2

Let  $A, B, C$  be a bipartite normalized LR triple with diameter  $d \geq 4$ . Let  $\mathfrak{X}$  be the tridiagonal space for  $A, B, C$ . By Theorem 1.2, the vector space  $\mathfrak{X}J$  has a basis

$$J, \quad AJ, \quad BJ, \quad ACBJ.$$

We represent the elements

$$CJ, \quad ABCJ, \quad BACJ, \quad BCAJ, \quad CABJ, \quad CBAJ$$

as a linear combination of the basis vectors. Below we give the coefficients of the linear combination.

For  $B_d(\mathbb{F}; t, \rho_0, \rho'_0, \rho''_0)$ :

	$J$	$AJ$	$BJ$	$ACBJ$
$CJ$	0	$\rho''_0$	$\frac{t}{\rho'_0}$	$\frac{\rho''_0(t-1)}{\rho'_0}$
$ABCJ$	0	$\frac{\rho_0 \rho''_0(t^{d/2}-1)}{t-1}$	0	$-\frac{\rho_0 \rho''_0}{\rho'_0}$
$BACJ$	0	$-\frac{\rho''_0}{\rho_0}$	$\frac{\rho''_0(t^{d/2+1}-1)}{t-1}$	$-\frac{\rho''_0 t}{\rho_0 \rho'_0}$
$BCAJ$	0	$\rho'_0$	$\frac{t}{\rho''_0}$	$t$
$CABJ$	0	0	$\frac{\rho''_0(t^{d/2+1}-1)}{t-1}$	$-\frac{\rho''_0 t}{\rho_0 \rho'_0}$
$CBAJ$	0	$\frac{\rho_0 \rho''_0(t^{d/2}-1)}{t-1}$	$-\frac{\rho_0}{\rho'_0}$	$-\frac{\rho_0 \rho'_0}{\rho''_0}$

For  $B_d(\mathbb{F}; 1, \rho_0, \rho'_0, \rho''_0)$ :

	$J$	$AJ$	$BJ$	$ACBJ$
$CJ$	0	$\rho''_0$	$\frac{1}{\rho'_0}$	0
$ABCJ$	0	$-\frac{d}{2\rho'_0}$	0	$\frac{1}{\rho'_0{}^2}$
$BACJ$	0	$-\frac{\rho''_0}{\rho_0}$	$\frac{\rho''_0(d+2)}{2}$	$\rho''_0{}^2$
$BCAJ$	0	$\rho'_0$	$\frac{1}{\rho''_0}$	1
$CABJ$	0	0	$\frac{\rho''_0(d+2)}{2}$	$\rho''_0{}^2$
$CBAJ$	0	$-\frac{d}{2\rho'_0}$	$-\frac{\rho_0}{\rho'_0}$	$\frac{1}{\rho'_0{}^2}$

By Theorem 1.2, the vector space  $\mathfrak{X}(I - J)$  has a basis

$$I - J, \quad A(I - J), \quad B(I - J), \quad ABC(I - J).$$

We represent the elements

$$C(I - J), \quad ACB(I - J), \quad BAC(I - J), \quad BCA(I - J), \quad CAB(I - J), \quad CBA(I - J)$$

as a linear combination of the basis vectors. Below we give the coefficients of the linear combination.

For  $B_d(\mathbb{F}; t, \rho_0, \rho'_0, \rho''_0)$ :

	$I - J$	$A(I - J)$	$B(I - J)$	$ABC(I - J)$
$C(I - J)$	0	$-\frac{\rho_0 \rho'_0}{t}$	$\rho'_0$	$-\frac{\rho_0(t-1)}{t}$
$ACB(I - J)$	0	$\frac{\rho_0 \rho''_0 (t^{d/2+1}-1)}{t(t-1)}$	0	$-\frac{\rho_0 \rho''_0}{\rho'_0 t}$
$BAC(I - J)$	0	$-\frac{\rho_0^2 \rho'_0 (t^{d/2}-1)}{t(t-1)}$	$\frac{\rho_0 \rho'_0 (t^{d/2}-1)}{t-1}$	$\frac{\rho_0^2}{t}$
$BCA(I - J)$	0	$\frac{\rho_0 \rho''_0 (t^{d/2}-1)}{t-1}$	$\rho''_0$	$-\frac{\rho_0 \rho''_0}{\rho'_0}$
$CAB(I - J)$	0	$-\frac{\rho_0^2 \rho'_0 (t^{d/2+1}-1)}{t(t-1)}$	$\frac{\rho_0 \rho'_0 (t^{d/2}-1)}{t-1}$	$\frac{\rho_0^2}{t}$
$CBA(I - J)$	0	0	$-\frac{\rho'_0}{\rho_0}$	1

For  $B_d(\mathbb{F}; 1, \rho_0, \rho'_0, \rho''_0)$ :

	$I - J$	$A(I - J)$	$B(I - J)$	$ABC(I - J)$
$C(I - J)$	0	$\frac{1}{\rho''_0}$	$\rho'_0$	0
$ACB(I - J)$	0	$-\frac{d+2}{2\rho'_0}$	0	$\frac{1}{\rho_0^2}$
$BAC(I - J)$	0	$\frac{d\rho_0}{2\rho''_0}$	$-\frac{d}{2\rho''_0}$	$\rho_0^2$
$BCA(I - J)$	0	$-\frac{d}{2\rho'_0}$	$\rho''_0$	$\frac{1}{\rho_0^2}$
$CAB(I - J)$	0	$\frac{\rho_0(d+2)}{2\rho''_0}$	$-\frac{d}{2\rho''_0}$	$\rho_0^2$
$CBA(I - J)$	0	0	$-\frac{\rho'_0}{\rho_0}$	1

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